

Quasimodular Hecke algebras and Hopf actions

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Abstract

Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. In this paper, we extend the theory of modular Hecke algebras due to Connes and Moscovici to define the algebra $\mathcal{Q}(\Gamma)$ of quasimodular Hecke operators of level Γ . Then, $\mathcal{Q}(\Gamma)$ carries an action of “the Hopf algebra \mathcal{H}_1 of codimension 1 foliations” that also acts on the modular Hecke algebra $\mathcal{A}(\Gamma)$ of Connes and Moscovici. However, in the case of quasimodular forms, we have several new operators acting on the quasimodular Hecke algebra $\mathcal{Q}(\Gamma)$. Further, for each $\sigma \in SL_2(\mathbb{Z})$, we introduce the collection $\mathcal{Q}_\sigma(\Gamma)$ of quasimodular Hecke operators of level Γ twisted by σ . Then, $\mathcal{Q}_\sigma(\Gamma)$ is a right $\mathcal{Q}(\Gamma)$ -module and is endowed with a pairing $(_, _) : \mathcal{Q}_\sigma(\Gamma) \otimes \mathcal{Q}_\sigma(\Gamma) \longrightarrow \mathcal{Q}_\sigma(\Gamma)$. We show that there is a “Hopf action” of a certain Hopf algebra \mathfrak{h}_1 on the pairing on $\mathcal{Q}_\sigma(\Gamma)$. Finally, for any $\sigma \in SL_2(\mathbb{Z})$, we consider operators acting between the levels of the graded module $\mathbb{Q}_\sigma(\Gamma) = \bigoplus_{m \in \mathbb{Z}} \mathcal{Q}_{\sigma(m)}(\Gamma)$, where $\sigma(m) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot \sigma$ for any $m \in \mathbb{Z}$. The pairing on $\mathcal{Q}_\sigma(\Gamma)$ can be extended to a graded pairing on $\mathbb{Q}_\sigma(\Gamma)$ and we show that there is a Hopf action of a larger Hopf algebra $\mathfrak{h}_\mathbb{Z} \supseteq \mathfrak{h}_1$ on the pairing on $\mathbb{Q}_\sigma(\Gamma)$.

Keywords: Modular Hecke algebras, Hopf actions

1 Introduction

Let $N \geq 1$ be an integer and let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. In [3], [4], Connes and Moscovici have introduced the “modular Hecke algebra” $\mathcal{A}(\Gamma)$ that combines the pointwise product on modular forms with the action of Hecke operators. Further, Connes and Moscovici have shown that the modular Hecke algebra $\mathcal{A}(\Gamma)$ carries an action of “the Hopf algebra \mathcal{H}_1 of codimension 1 foliations”. The Hopf algebra \mathcal{H}_1 is part of a larger family of Hopf algebras $\{\mathcal{H}_n | n \geq 1\}$ defined in [2]. The objective of this paper is to introduce and study quasimodular Hecke algebras $\mathcal{Q}(\Gamma)$ that similarly combine the pointwise product on quasimodular forms with the action of Hecke operators. We will see that the quasimodular Hecke algebra $\mathcal{Q}(\Gamma)$ carries several other operators in addition to an action of \mathcal{H}_1 . Further, we will also study the collection $\mathcal{Q}_\sigma(\Gamma)$ of quasimodular Hecke operators twisted by some $\sigma \in SL_2(\mathbb{Z})$. The latter is a generalization of our theory of twisted modular Hecke operators introduced in [1].

We now describe the paper in detail. In Section 2, we briefly recall the notion of modular Hecke algebras of Connes and Moscovici [3], [4]. We let \mathcal{QM} be the “quasimodular tower”, i.e., \mathcal{QM} is the colimit over all N of the spaces $\mathcal{QM}(\Gamma(N))$ of quasimodular forms of level $\Gamma(N)$ (see (2.8)). We define a quasimodular Hecke operator of level Γ to be a function of finite support from $\Gamma \backslash GL_2^+(\mathbb{Q})$ to the quasimodular tower \mathcal{QM} satisfying a certain covariance condition (see Definition 2.4). We then show that the collection $\mathcal{Q}(\Gamma)$ of quasimodular Hecke

operators of level Γ carries an algebra structure $(\mathcal{Q}(\Gamma), *)$ by considering a convolution product over cosets of Γ in $GL_2^+(\mathbb{Q})$. Further, the modular Hecke algebra of Connes and Moscovici embeds naturally as a subalgebra of $\mathcal{Q}(\Gamma)$. We also show that the quasimodular Hecke operators of level Γ act on quasimodular forms of level Γ , i.e., $\mathcal{QM}(\Gamma)$ is a left $\mathcal{Q}(\Gamma)$ -module. In this section, we will also define a second algebra structure $(\mathcal{Q}(\Gamma), *^r)$ on $\mathcal{Q}(\Gamma)$ by considering the convolution product over cosets of Γ in $SL_2(\mathbb{Z})$. When we consider $\mathcal{Q}(\Gamma)$ as an algebra equipped with this latter product $*^r$, it will be denoted by $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), *^r)$.

In Section 3, we define Lie algebra and Hopf algebra actions on $\mathcal{Q}(\Gamma)$. Given a quasimodular form $f \in \mathcal{QM}(\Gamma)$ of level Γ , it is well known that we can write f as a sum

$$f = \sum_{i=0}^s a_i(f) \cdot G_2^i \quad (1.1)$$

where the coefficients $a_i(f)$ are modular forms of level Γ and G_2 is the classical Eisenstein series of weight 2. Therefore, we can consider two different sets of operators on the quasimodular tower \mathcal{QM} : those which act on the powers of G_2 appearing in the expression for f and those which act on the modular coefficients $a_i(f)$. The collection of operators acting on the modular coefficients $a_i(f)$ are studied in Section 3.2. These induce on $\mathcal{Q}(\Gamma)$ analogues of operators acting on the modular Hecke algebra $\mathcal{A}(\Gamma)$ of Connes and Moscovici and we show that $\mathcal{Q}(\Gamma)$ carries an action of the same Hopf algebra \mathcal{H}_1 of codimension 1 foliations that acts on $\mathcal{A}(\Gamma)$. On the other hand, by considering operators on \mathcal{QM} that act on the powers of G_2 appearing in (1.1), we are able to define additional operators D , $\{T_k^l\}_{k \geq 1, l \geq 0}$ and $\{\phi^{(m)}\}_{m \geq 1}$ on $\mathcal{Q}(\Gamma)$ (see Section 3.1). Further, we show that these operators satisfy the following commutator relations:

$$\begin{aligned} [T_k^l, T_{k'}^{l'}] &= (k' - k) T_{k+k'-2}^{l+l'} \\ [D, \phi^{(m)}] &= 0 \quad [T_k^l, \phi^{(m)}] = 0 \quad [\phi^{(m)}, \phi^{(m')}] = 0 \\ [T_k^l, D] &= \frac{5}{24}(k-1)T_{k-1}^{l+1} - \frac{1}{2}(k-3)T_{k+1}^l \end{aligned} \quad (1.2)$$

We then consider the Lie algebra \mathcal{L} generated by the symbols D , $\{T_k^l\}_{k \geq 1, l \geq 0}$, $\{\phi^{(m)}\}_{m \geq 1}$ satisfying the commutator relations in (1.2). Then, there is a Lie action of \mathcal{L} on $\mathcal{Q}(\Gamma)$. Finally, let \mathcal{H} be the Hopf algebra given by the universal enveloping algebra $\mathcal{U}(\mathcal{L})$ of \mathcal{L} . Then, we show that \mathcal{H} has a Hopf action with respect to the product $*^r$ on $\mathcal{Q}(\Gamma)$ and this action captures the operators D , $\{T_k^l\}_{k \geq 1, l \geq 0}$ and $\{\phi^{(m)}\}_{m \geq 1}$ on $\mathcal{Q}(\Gamma)$. In other words, \mathcal{H} acts on $\mathcal{Q}(\Gamma)$ such that:

$$h(F^1 *^r F^2) = \sum h_{(1)}(F^1) *^r h_{(2)}(F^2) \quad \forall h \in \mathcal{H}, F^1, F^2 \in \mathcal{Q}(\Gamma) \quad (1.3)$$

where the coproduct $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ is given by $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for any $h \in \mathcal{H}$.

In Section 4, we develop the theory of twisted quasimodular Hecke operators. For any $\sigma \in SL_2(\mathbb{Z})$, we define in Section 4.1 the collection $\mathcal{Q}_\sigma(\Gamma)$ of quasimodular Hecke operators of level Γ twisted by σ . When $\sigma = 1$, this reduces to the original definition of $\mathcal{Q}(\Gamma)$. In general, $\mathcal{Q}_\sigma(\Gamma)$ is not an algebra but we show that $\mathcal{Q}_\sigma(\Gamma)$ carries a pairing:

$$(_, _) : \mathcal{Q}_\sigma(\Gamma) \otimes \mathcal{Q}_\sigma(\Gamma) \rightarrow \mathcal{Q}_\sigma(\Gamma) \quad (1.4)$$

Further, we show that $\mathcal{Q}_\sigma(\Gamma)$ may be equipped with the structure of a right $\mathcal{Q}(\Gamma)$ -module. We can also extend the action of the Hopf algebra \mathcal{H}_1 of codimension 1 foliations to $\mathcal{Q}_\sigma(\Gamma)$. In fact, we show that \mathcal{H}_1 has a ‘‘Hopf action’’ on the right $\mathcal{Q}(\Gamma)$ module $\mathcal{Q}_\sigma(\Gamma)$, i.e.,

$$h(F^1 * F^2) = \sum h_{(1)}(F^1) * h_{(2)}(F^2) \quad \forall h \in \mathcal{H}_1, F^1 \in \mathcal{Q}_\sigma(\Gamma), F^2 \in \mathcal{Q}(\Gamma) \quad (1.5)$$

where the coproduct $\Delta : \mathcal{H}_1 \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_1$ is given by $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for any $h \in \mathcal{H}_1$. We recall from [3] that \mathcal{H}_1 is equal as an algebra to the universal enveloping algebra of the Lie algebra \mathcal{L}_1 with generators $X, Y, \{\delta_n\}_{n \geq 1}$ satisfying the following relations:

$$[Y, X] = X \quad [X, \delta_n] = \delta_{n+1} \quad [Y, \delta_n] = n\delta_n \quad [\delta_k, \delta_l] = 0 \quad \forall k, l, n \geq 1 \quad (1.6)$$

Then, we can consider the smaller Lie algebra $\mathfrak{l}_1 \subseteq \mathcal{L}_1$ with two generators X, Y satisfying $[Y, X] = X$. If we let \mathfrak{h}_1 be the Hopf algebra that is the universal enveloping algebra of \mathfrak{l}_1 , we show that the pairing in (1.4) on $\mathcal{Q}_\sigma(\Gamma)$ carries a ‘‘Hopf action’’ of \mathfrak{h}_1 . In other words, we have:

$$h(F^1, F^2) = \sum (h_{(1)}(F^1), h_{(2)}(F^2)) \quad \forall h \in \mathfrak{h}_1, F^1, F^2 \in \mathcal{Q}_\sigma(\Gamma) \quad (1.7)$$

where the coproduct $\Delta : \mathfrak{h}_1 \longrightarrow \mathfrak{h}_1 \otimes \mathfrak{h}_1$ is given by $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for any $h \in \mathfrak{h}_1$. In Section 4.2, we consider operators between the modules $\mathcal{Q}_\sigma(\Gamma)$ as σ varies over $SL_2(\mathbb{Z})$. More precisely, for any $\tau, \sigma \in SL_2(\mathbb{Z})$, we define a morphism:

$$X_\tau : \mathcal{Q}_\sigma(\Gamma) \longrightarrow \mathcal{Q}_{\tau\sigma}(\Gamma) \quad (1.8)$$

In particular, this gives us operators acting between the levels of the graded module

$$\mathbb{Q}_\sigma(\Gamma) = \bigoplus_{m \in \mathbb{Z}} \mathcal{Q}_{\sigma(m)}(\Gamma) \quad (1.9)$$

where for any $\sigma \in SL_2(\mathbb{Z})$, we set $\sigma(m) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot \sigma$. Further, we generalize the pairing on $\mathcal{Q}_\sigma(\Gamma)$ in (1.4) to a pairing:

$$(_, _) : \mathcal{Q}_{\tau_1\sigma}(\Gamma) \otimes \mathcal{Q}_{\tau_2\sigma}(\Gamma) \longrightarrow \mathcal{Q}_{\tau_1\tau_2\sigma}(\Gamma) \quad (1.10)$$

where τ_1, τ_2 are commuting matrices in $SL_2(\mathbb{Z})$. In particular, (1.10) gives us a pairing $\mathcal{Q}_{\sigma(m)}(\Gamma) \otimes \mathcal{Q}_{\sigma(n)}(\Gamma) \longrightarrow \mathcal{Q}_{\sigma(m+n)}(\Gamma)$, $\forall m, n \in \mathbb{Z}$ and hence a pairing on the tower $\mathbb{Q}_\sigma(\Gamma)$. Finally, we consider the Lie algebra $\mathfrak{l}_\mathbb{Z} \supseteq \mathfrak{l}_1$ with generators $\{Z, X_n | n \in \mathbb{Z}\}$ satisfying the following commutator relations:

$$[Z, X_n] = (n+1)X_n \quad [X_n, X_{n'}] = 0 \quad \forall n, n' \in \mathbb{Z} \quad (1.11)$$

Then, if we let $\mathfrak{h}_\mathbb{Z}$ be the Hopf algebra that is the universal enveloping algebra of $\mathfrak{l}_\mathbb{Z}$, we show that $\mathfrak{h}_\mathbb{Z}$ has a Hopf action on the pairing on $\mathbb{Q}_\sigma(\Gamma)$. In other words, for any $F^1, F^2 \in \mathbb{Q}_\sigma(\Gamma)$, we have

$$h(F^1, F^2) = \sum (h_{(1)}(F^1), h_{(2)}(F^2)) \quad \forall h \in \mathfrak{h}_\mathbb{Z} \quad (1.12)$$

where the coproduct $\Delta : \mathfrak{h}_\mathbb{Z} \longrightarrow \mathfrak{h}_\mathbb{Z} \otimes \mathfrak{h}_\mathbb{Z}$ is defined by setting $\Delta(h) := \sum h_{(1)} \otimes h_{(2)}$ for each $h \in \mathfrak{h}_\mathbb{Z}$.

2 The Quasimodular Hecke algebra

We begin this section by briefly recalling the notion of quasimodular forms. The notion of quasimodular forms is due to Kaneko and Zagier [5]. The theory has been further developed in Zagier [7]. For an introduction to the basic theory of quasimodular forms, we refer the reader to the exposition of Royer [6].

Throughout, let $\mathbb{H} \subseteq \mathbb{C}$ be the upper half plane. Then, there is a well known action of $SL_2(\mathbb{Z})$ on \mathbb{H} :

$$z \mapsto \frac{az+b}{cz+d} \quad \forall z \in \mathbb{H}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \quad (2.1)$$

For any $N \geq 1$, we denote by $\Gamma(N)$ the following principal congruence subgroup of $SL_2(\mathbb{Z})$:

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\} \quad (2.2)$$

In particular, $\Gamma(1) = SL_2(\mathbb{Z})$. We are now ready to define quasimodular forms.

Definition 2.1. *Let $f : \mathbb{H} \rightarrow \mathbb{C}$ be a holomorphic function and let $N \geq 1$, $k, s \geq 0$ be integers. Then, the function f is a quasimodular form of level N , weight k and depth s if there exist holomorphic functions $f_0, f_1, \dots, f_s : \mathbb{H} \rightarrow \mathbb{C}$ with $f_s \neq 0$ such that:*

$$(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) = \sum_{j=0}^s f_j(z) \left(\frac{c}{cz + d}\right)^j \quad (2.3)$$

for any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$. The collection of quasimodular forms of level N , weight k and depth s will be denoted by $\mathcal{QM}_k^s(\Gamma(N))$. By convention, we let the zero function $0 \in \mathcal{QM}_k^0(\Gamma(N))$ for every $k \geq 0$, $N \geq 1$.

More generally, for any holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ and any matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$, we define:

$$(f|_k \alpha)(z) := (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) \quad \forall k \geq 0 \quad (2.4)$$

Then, we can say that f is quasimodular of level N , weight k and depth s if there exist holomorphic functions $f_0, f_1, \dots, f_s : \mathbb{H} \rightarrow \mathbb{C}$ with $f_s \neq 0$ such that:

$$(f|_k \gamma)(z) = \sum_{j=0}^s f_j(z) \left(\frac{c}{cz + d}\right)^j \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N) \quad (2.5)$$

When the integer k is clear from context, we write $f|_k \alpha$ simply as $f|\alpha$ for any $\alpha \in GL_2^+(\mathbb{Q})$. Also, it is clear that we have a product:

$$\mathcal{QM}_k^s(\Gamma(N)) \otimes \mathcal{QM}_l^t(\Gamma(N)) \rightarrow \mathcal{QM}_{k+l}^{s+t}(\Gamma(N)) \quad (2.6)$$

on quasi-modular forms. For any $N \geq 1$, we now define:

$$\mathcal{QM}(\Gamma(N)) := \bigoplus_{s=0}^{\infty} \bigoplus_{k=0}^{\infty} \mathcal{QM}_k^s(\Gamma(N)) \quad (2.7)$$

We now consider the direct limit:

$$\mathcal{QM} := \varinjlim_{N \geq 1} \mathcal{QM}(\Gamma(N)) \quad (2.8)$$

which we will refer to as the quasimodular tower. Additionally, for any $k \geq 0$ and $N \geq 1$, we let $\mathcal{M}_k(\Gamma(N))$ denote the collection of usual modular forms of weight k and level N . Then, we can define the modular tower \mathcal{M} :

$$\mathcal{M} := \varinjlim_{N \geq 1} \mathcal{M}(\Gamma(N)) \quad \mathcal{M}(\Gamma(N)) := \bigoplus_{k=0}^{\infty} \mathcal{M}_k(\Gamma(N)) \quad (2.9)$$

We now recall the modular Hecke algebra of Connes and Moscovici [3].

Definition 2.2. (see [3, § 1]) Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. A modular Hecke operator of level Γ is a function of finite support

$$F : \Gamma \backslash GL_2^+(\mathbb{Q}) \longrightarrow \mathcal{M} \quad \Gamma\alpha \mapsto F_\alpha \quad (2.10)$$

such that for any $\gamma \in \Gamma$, we have:

$$F_{\alpha\gamma} = F_\alpha|_\gamma \quad (2.11)$$

The collection of all modular Hecke operators of level Γ will be denoted by $\mathcal{A}(\Gamma)$.

Our first aim is to define a quasimodular Hecke algebra $\mathcal{Q}(\Gamma)$ analogous to the modular Hecke algebra $\mathcal{A}(\Gamma)$ of Connes and Moscovici. For this, we recall the structure theorem for quasimodular forms, proved by Kaneko and Zagier [5].

Theorem 2.3. (see [5, § 1, Proposition 1.]) Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. For any even number $K \geq 2$, let G_K denote the classical Eisenstein series of weight K :

$$G_K(z) := -\frac{B_K}{2K} + \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{K-1} \right) e^{2\pi i n z} \quad (2.12)$$

where B_K is the K -th Bernoulli number and $z \in \mathbb{H}$. Then, every quasimodular form in $\mathcal{QM}(\Gamma)$ can be written uniquely as a polynomial in G_2 with coefficients in $\mathcal{M}(\Gamma)$. More precisely, for any quasimodular form $f \in \mathcal{QM}_k^s(\Gamma)$, there exist functions $a_0(f)$, $a_1(f)$, ..., $a_s(f)$ such that:

$$f = \sum_{i=0}^s a_i(f) G_2^i \quad (2.13)$$

where $a_i(f) \in \mathcal{M}_{k-2i}(\Gamma)$ is a modular form of weight $k-2i$ and level Γ for each $0 \leq i \leq s$.

We now consider a quasimodular form $f \in \mathcal{QM}$. For sake of definiteness, we may assume that $f \in \mathcal{QM}_k^s(\Gamma(N))$, i.e. f is a quasimodular form of level N , weight k and depth s . We now define an operation on \mathcal{QM} by setting:

$$f||\alpha = \sum_{i=0}^s (a_i(f)|_{k-2i}\alpha) G_2^i \quad \forall \alpha \in GL_2^+(\mathbb{Q}) \quad (2.14)$$

where $\{a_i(f) \in \mathcal{M}_{k-2i}(\Gamma(N))\}_{0 \leq i \leq s}$ is the collection of modular forms determining $f = \sum_{i=0}^s a_i(f) G_2^i$ as in Theorem 2.3. We know that for any $\alpha \in GL_2^+(\mathbb{Q})$, each $(a_i(f)|_{k-2i}\alpha)$ is an element of the modular tower \mathcal{M} . This shows that $f||\alpha = \sum_{i=0}^s (a_i(f)|_{k-2i}\alpha) G_2^i \in \mathcal{QM}$. However, we note that for arbitrary $\alpha \in GL_2^+(\mathbb{Q})$ and $a_i(f) \in \mathcal{M}_{k-2i}(\Gamma(N))$, it is not necessary that $(a_i(f)|_{k-2i}\alpha) \in \mathcal{M}_{k-2i}(\Gamma(N))$. In other words, the operation defined in (2.14) on the quasimodular tower \mathcal{QM} does not descend to an endomorphism on each $\mathcal{QM}_k^s(\Gamma(N))$. From the expression in (2.14), it is also clear that:

$$(f \cdot g)||\alpha = (f||\alpha) \cdot (g||\alpha) \quad f||(\alpha \cdot \beta) = (f||\alpha)||\beta \quad \forall f, g \in \mathcal{QM}, \alpha, \beta \in GL_2^+(\mathbb{Q}) \quad (2.15)$$

We are now ready to define the quasimodular Hecke operators.

Definition 2.4. Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup. A quasimodular Hecke operator of level Γ is a function of finite support:

$$F : \Gamma \backslash GL_2^+(\mathbb{Q}) \longrightarrow \mathcal{QM} \quad \Gamma\alpha \mapsto F_\alpha \quad (2.16)$$

such that for any $\gamma \in \Gamma$, we have:

$$F_{\alpha\gamma} = F_\alpha || \gamma \quad (2.17)$$

The collection of all quasimodular Hecke operators of level Γ will be denoted by $\mathcal{Q}(\Gamma)$.

We will now introduce the product structure on $\mathcal{Q}(\Gamma)$. In fact, we will introduce two separate product structures $(\mathcal{Q}(\Gamma), *)$ and $(\mathcal{Q}(\Gamma), *^r)$ on $\mathcal{Q}(\Gamma)$.

Proposition 2.5. (a) Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup and let $\mathcal{Q}(\Gamma)$ be the collection of quasimodular Hecke operators of level Γ . Then, the product defined by:

$$(F * G)_\alpha := \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta \cdot (G_{\alpha\beta^{-1}} || \beta) \quad \forall \alpha \in GL_2^+(\mathbb{Q}) \quad (2.18)$$

for all $F, G \in \mathcal{Q}(\Gamma)$ makes $\mathcal{Q}(\Gamma)$ into an associative algebra.

(b) Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup and let $\mathcal{Q}(\Gamma)$ be the collection of quasimodular Hecke operators of level Γ . Then, the product defined by:

$$(F *^r G)_\alpha := \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} F_\beta \cdot (G_{\alpha\beta^{-1}} || \beta) \quad \forall \alpha \in GL_2^+(\mathbb{Q}) \quad (2.19)$$

for all $F, G \in \mathcal{Q}(\Gamma)$ makes $\mathcal{Q}(\Gamma)$ into an associative algebra which we denote by $\mathcal{Q}^r(\Gamma)$.

Proof. (a) We need to check that the product in (2.18) is associative. First of all, we note that the expression in (2.18) can be rewritten as:

$$(F * G)_\alpha = \sum_{\alpha_2 \alpha_1 = \alpha} F_{\alpha_1} \cdot G_{\alpha_2} || \alpha_1 \quad \forall \alpha \in GL_2^+(\mathbb{Q}) \quad (2.20)$$

where the sum in (2.20) is taken over all pairs (α_1, α_2) with $\alpha_2 \alpha_1 = \alpha$ modulo the following equivalence relation:

$$(\alpha_1, \alpha_2) \sim (\gamma \alpha_1, \alpha_2 \gamma^{-1}) \quad \forall \gamma \in \Gamma \quad (2.21)$$

Hence, for $F, G, H \in \mathcal{Q}(\Gamma)$, we can write:

$$\begin{aligned} (F * (G * H))_\alpha &= \sum_{\alpha'_2 \alpha_1 = \alpha} F_{\alpha_1} \cdot (G * H)_{\alpha'_2} || \alpha_1 \\ &= \sum_{\alpha'_2 \alpha_1 = \alpha} F_{\alpha_1} \cdot (\sum_{\alpha_3 \alpha_2 = \alpha'_2} G_{\alpha_2} \cdot H_{\alpha_3} || \alpha_2) || \alpha_1 \\ &= \sum_{\alpha_3 \alpha_2 \alpha_1 = \alpha} F_{\alpha_1} \cdot (G_{\alpha_2} || \alpha_1) \cdot (H_{\alpha_3} || \alpha_2 \alpha_1) \end{aligned} \quad (2.22)$$

where the sum in (2.22) is taken over all triples $(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_3 \alpha_2 \alpha_1 = \alpha$ modulo the following equivalence relation:

$$(\alpha_1, \alpha_2, \alpha_3) \sim (\gamma \alpha_1, \gamma' \alpha_2 \gamma'^{-1}, \alpha_3 \gamma'^{-1}) \quad \forall \gamma, \gamma' \in \Gamma \quad (2.23)$$

On the other hand, we have

$$\begin{aligned} ((F * G) * H)_\alpha &= \sum_{\alpha_3 \alpha'_2 = \alpha} (F * G)_{\alpha'_2} \cdot H_{\alpha_3} || \alpha''_2 \\ &= \sum_{\alpha_3 \alpha'_2 = \alpha} (\sum_{\alpha_2 \alpha_1 = \alpha'_2} F_{\alpha_1} \cdot G_{\alpha_2} || \alpha_1) \cdot H_{\alpha_3} || \alpha''_2 \\ &= \sum_{\alpha_3 \alpha_2 \alpha_1 = \alpha} F_{\alpha_1} \cdot (G_{\alpha_2} || \alpha_1) \cdot (H_{\alpha_3} || \alpha_2 \alpha_1) \end{aligned} \quad (2.24)$$

where the sum in (2.24) is taken over all triples $(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_3 \alpha_2 \alpha_1 = \alpha$ modulo the equivalence relation in (2.23). From (2.22) and (2.24) the result follows. \square

We know that modular forms are quasimodular forms of depth 0, i.e., for any $k \geq 0$, $N \geq 1$, we have $\mathcal{M}_k(\Gamma(N)) = \mathcal{QM}_k^0(\Gamma(N))$. It follows that the modular tower \mathcal{M} defined in (2.9) embeds into the quasimodular tower \mathcal{QM} defined in (2.8). We are now ready to show that the modular Hecke algebra $\mathcal{A}(\Gamma)$ of Connes and Moscovici embeds into the quasimodular Hecke algebra $\mathcal{Q}(\Gamma)$ for any congruence subgroup $\Gamma = \Gamma(N)$.

Proposition 2.6. *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. Let $\mathcal{A}(\Gamma)$ be the modular Hecke algebra of level Γ as defined in Definition 2.2 and let $\mathcal{Q}(\Gamma)$ be the quasimodular Hecke algebra of level Γ as defined in Definition 2.4. Then, there is a natural embedding of algebras $\mathcal{A}(\Gamma) \hookrightarrow \mathcal{Q}(\Gamma)$.*

Proof. For any $\alpha \in GL_2^+(\mathbb{Q})$ and any $f \in \mathcal{QM}_k^s(\Gamma)$, we consider the operation $f \mapsto f||\alpha$ as defined in (2.14):

$$f||\alpha = \sum_{i=0}^s (a_i(f)|_{k-2i}\alpha) G_2^i \in \mathcal{QM} \quad (2.25)$$

In particular, if $f \in \mathcal{M}_k(\Gamma) = \mathcal{QM}_k^0(\Gamma)$ is a modular form, it follows from (2.25) that:

$$f||\alpha = a_0(f)|_k\alpha = f|_k\alpha = f|\alpha \in \mathcal{M} \quad (2.26)$$

Hence, using the embedding of \mathcal{M} in \mathcal{QM} , it follows from (2.11) in the definition of $\mathcal{A}(\Gamma)$ and from (2.17) in the definition of $\mathcal{Q}(\Gamma)$ that we have an embedding $\mathcal{A}(\Gamma) \hookrightarrow \mathcal{Q}(\Gamma)$ of modules. Further, we recall from [3, § 1] that the product on $\mathcal{A}(\Gamma)$ is given by:

$$(F * G)_\alpha := \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta \cdot (G_{\alpha\beta^{-1}}|\beta) \quad \forall \alpha \in GL_2^+(\mathbb{Q}), F, G \in \mathcal{A}(\Gamma) \quad (2.27)$$

Comparing (2.27) with the product on $\mathcal{Q}(\Gamma)$ described in (2.18) and using (2.26) it follows that $\mathcal{A}(\Gamma) \hookrightarrow \mathcal{Q}(\Gamma)$ is an embedding of algebras. \square

We end this section by describing the action of the algebra $\mathcal{Q}(\Gamma)$ on $\mathcal{QM}(\Gamma)$.

Proposition 2.7. *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup and let $\mathcal{Q}(\Gamma)$ be the algebra of quasimodular Hecke operators of level Γ . Then, for any element $f \in \mathcal{QM}(\Gamma)$ the action of $\mathcal{Q}(\Gamma)$ defined by:*

$$F * f := \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta \cdot f||\beta \quad \forall F \in \mathcal{Q}(\Gamma) \quad (2.28)$$

makes $\mathcal{QM}(\Gamma)$ into a left module over $\mathcal{Q}(\Gamma)$.

Proof. It is easy to check that the right hand side of (2.28) is independent of the choice of coset representatives. Further, since $F \in \mathcal{Q}(\Gamma)$ is a function of finite support, we can choose finitely many coset representatives $\{\beta_1, \beta_2, \dots, \beta_n\}$ such that

$$F * f = \sum_{j=1}^n F_{\beta_j} \cdot f||\beta_j \quad (2.29)$$

It suffices to consider the case $f \in \mathcal{QM}_k^s(\Gamma)$ for some weight k and depth s . Then, we can express f as a sum:

$$f = \sum_{i=0}^s a_i(f) G_2^i \quad (2.30)$$

where each $a_i(f) \in \mathcal{M}_{k-2i}(\Gamma)$. Similarly, for any $\beta \in GL_2^+(\mathbb{Q})$, we can express F_β as a finite sum:

$$F_\beta = \sum_{r=0}^{t_\beta} a_{\beta r}(F_\beta) \cdot G_2^r \quad (2.31)$$

with each $a_{\beta r}(F_\beta) \in \mathcal{M}$. In particular, we let $t = \max\{t_{\beta_1}, t_{\beta_2}, \dots, t_{\beta_n}\}$ and we can now write:

$$F_{\beta_j} = \sum_{r=0}^t a_{\beta_j r}(F_{\beta_j}) \cdot G_2^r \quad (2.32)$$

by adding appropriately many terms with zero coefficients in the expression for each F_{β_j} . Further, for any $\gamma \in \Gamma$, we know that $F_{\beta_j \gamma} = F_{\beta_j}||\gamma = \sum_{r=0}^t (a_{\beta_j r}(F_{\beta_j})|\gamma) \cdot G_2^r$. In other words, we have, for each j :

$$F_{\beta_j \gamma} = \sum_{r=0}^t a_{\beta_j \gamma r}(F_{\beta_j \gamma}) \cdot G_2^r \quad a_{\beta_j \gamma r}(F_{\beta_j \gamma}) = (a_{\beta_j r}(F_{\beta_j})|\gamma) \quad (2.33)$$

The sum in (2.29) can now be expressed as:

$$F * f := \sum_{j=1}^n F_{\beta_j} \cdot f||\beta_j = \sum_{i=0}^s \sum_{r=0}^t \sum_{j=1}^n a_{\beta_j r}(F_{\beta_j}) \cdot (a_i(f)|\beta_j) \cdot G_2^{r+i} \quad (2.34)$$

For any i, r , we now set:

$$A_{ir}(F, f) := \sum_{j=1}^n a_{\beta_j r}(F_{\beta_j}) \cdot (a_i(f)|\beta_j) \quad (2.35)$$

Again, it is easy to see that the sum $A_{ir}(F, f)$ in (2.35) does not depend on the choice of the coset representatives $\{\beta_1, \beta_2, \dots, \beta_n\}$. Then, for any $\gamma \in \Gamma$, we have:

$$A_{ir}(F, f)|\gamma = \sum_{j=1}^n (a_{\beta_j r}(F_{\beta_j})|\gamma) \cdot (a_i(f)|\beta_j \gamma) = \sum_{j=1}^n a_{\beta_j \gamma r}(F_{\beta_j \gamma}) \cdot (a_i(f)|\beta_j \gamma) = A_{ir}(F, f) \quad (2.36)$$

where the last equality in (2.36) follows from the fact that $\{\beta_1 \gamma, \beta_2 \gamma, \dots, \beta_n \gamma\}$ is another collection of distinct cosets representatives of Γ in $GL_2^+(\mathbb{Q})$. From (2.36), we note that each $A_{ir}(F, f) \in \mathcal{M}(\Gamma)$. Then, the sum:

$$F * f = \sum_{i=0}^s \sum_{r=0}^t A_{ir}(F, f) \cdot G_2^{i+r} \quad (2.37)$$

is an element of $\mathcal{QM}(\Gamma)$. Hence, $\mathcal{QM}(\Gamma)$ is a left module over $\mathcal{Q}(\Gamma)$. \square

3 The Lie algebra and Hopf algebra actions on $\mathcal{Q}(\Gamma)$

Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. In this section, we will describe two different sets of operators on the collection $\mathcal{Q}(\Gamma)$ of quasimodular Hecke operators of level Γ . Given a quasimodular form $f \in \mathcal{QM}(\Gamma)$ of level Γ , we have mentioned in the last section that f can be expressed as a finite sum:

$$f = \sum_{i=0}^s a_i(f) \cdot G_2^i \quad (3.1)$$

where G_2 is the classical Eisenstein series of weight 2 and each $a_i(f)$ is a modular form of level Γ . Then in Section 3.1, we consider operators on the quasimodular tower that act on the powers of G_2 appearing in (3.1). These induce operators $D, \{T_k^l\}_{k \geq 1, l \geq 0}$ on the collection $\mathcal{Q}(\Gamma)$ of quasimodular Hecke operators of level Γ . In order to understand the action of these operators on products of elements in $\mathcal{Q}(\Gamma)$, we also need to define extra operators $\{\phi^{(m)}\}_{m \geq 1}$. Finally, we show that these operators may all be described in terms of a Hopf algebra \mathcal{H} with a ‘‘Hopf action’’ on $\mathcal{Q}^r(\Gamma)$, i.e.,

$$h(F^1 *^r F^2) = \sum h_{(1)}(F^1) *^r h_{(2)}(F^2) \quad \forall h \in \mathcal{H}, F^1, F^2 \in \mathcal{Q}^r(\Gamma) \quad (3.2)$$

where the coproduct $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ is given by $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for any $h \in \mathcal{H}$. In Section 3.2, we consider operators on the quasimodular tower \mathcal{QM} that act on the modular coefficients $a_i(f)$ appearing in (3.1). These induce on $\mathcal{Q}(\Gamma)$ analogues of operators acting on the modular Hecke algebra $\mathcal{A}(\Gamma)$ of Connes and Moscovici [3]. Then, we show that $\mathcal{Q}(\Gamma)$ carries a Hopf action of the same Hopf algebra \mathcal{H}_1 of codimension 1 foliations that acts on $\mathcal{A}(\Gamma)$.

3.1 The operators $D, \{T_k^l\}$ and $\{\phi^{(m)}\}$ on $\mathcal{Q}(\Gamma)$

For any even number $K \geq 2$, let G_K be the classical Eisenstein series of weight K as in (2.12). Since G_2 is a quasimodular form, i.e., $G_2 \in \mathcal{QM}$, its derivative $G_2' \in \mathcal{QM}$. Further, it is well known that:

$$G_2' = \frac{5\pi i}{3} G_4 - 4\pi i G_2^2 \quad (3.3)$$

where G_4 is the Eisenstein series of weight 4 (which is a modular form). For our purposes, it will be convenient to write:

$$G_2' = \sum_{j=0}^2 g_j G_2^j \quad (3.4)$$

with each g_j a modular form. From (3.3), it follows that:

$$g_0 = \frac{5\pi i}{3} G_4 \quad g_1 = 0 \quad g_2 = -4\pi i \quad (3.5)$$

We are now ready to define the operators D and $\{W_k\}_{k \geq 1}$ on \mathcal{QM} . The first operator D differentiates the powers of G_2 :

$$\begin{aligned} D : \mathcal{QM} &\rightarrow \mathcal{QM} \\ f = \sum_{i=0}^s a_i(f) G_2^i &\mapsto -\frac{1}{8\pi i} \left(\sum_{i=0}^s i a_i(f) G_2^{i-1} \cdot G_2' \right) \\ &= -\frac{1}{8\pi i} \sum_{i=0}^s \sum_{j=0}^2 i a_i(f) g_j G_2^{i+j-1} \end{aligned} \quad (3.6)$$

The operators $\{W_k\}_{k \geq 1}$ are ‘‘weight operators’’ and W_k also steps up the power of G_2 by $k - 2$. We set:

$$W_k : \mathcal{QM} \rightarrow \mathcal{QM} \quad f = \sum_{i=0}^s a_i(f) G_2^i \mapsto \sum_{i=0}^s i a_i(f) G_2^{i+k-2} \quad (3.7)$$

From the definitions in (3.6) and (3.7), we can easily check that D and W_k are derivations on \mathcal{QM} . Finally, for any $\alpha \in GL_2^+(\mathbb{Q})$ and any integer $m \geq 1$, we set

$$\nu_\alpha^{(m)} = -\frac{5}{24} (G_4^m | \alpha - G_4^m) \quad (3.8)$$

Lemma 3.1. (a) Let $f \in \mathcal{QM}$ be an element of the quasimodular tower and $\alpha \in GL_2^+(\mathbb{Q})$. Then, the operator D satisfies:

$$D(f)||\alpha = D(f||\alpha) + \nu_\alpha^{(1)} \cdot (W_1(f)||\alpha) \quad (3.9)$$

where, using (3.8), we know that $\nu_\alpha^{(1)}$ is given by:

$$\nu_\alpha^{(1)} := -\frac{1}{8\pi i}(g_0|\alpha - g_0) = -\frac{5}{24}(G_4|\alpha - G_4) \quad \forall \alpha \in GL_2^+(\mathbb{Q}) \quad (3.10)$$

(b) For $f \in \mathcal{QM}$ and $\alpha \in GL_2^+(\mathbb{Q})$, each operator W_k , $k \geq 1$ satisfies:

$$W_k(f)||\alpha = W_k(f||\alpha) \quad (3.11)$$

Proof. We start by proving part (a). For the sake of definiteness, we assume that $f = \sum_{i=0}^s a_i(f)G_2^i$ with each $a_i(f) \in \mathcal{M}$. For $\alpha \in GL_2^+(\mathbb{Q})$, it follows from (3.6) that:

$$\begin{aligned} D(f)||\alpha &= -\frac{1}{8\pi i} \left(\sum_i \sum_j i a_i(f) g_j G_2^{i+j-1} \right) ||\alpha & D(f||\alpha) &= D \left(\sum_i (a_i(f)||\alpha) G_2^i \right) \\ &= -\frac{1}{8\pi i} \sum_i \sum_j i (a_i(f)||\alpha) (g_j||\alpha) G_2^{i+j-1} & &= -\frac{1}{8\pi i} \sum_i \sum_j i (a_i(f)||\alpha) g_j G_2^{i+j-1} \end{aligned} \quad (3.12)$$

From (3.12) it follows that:

$$D(f)||\alpha - D(f||\alpha) = -\frac{1}{8\pi i} \sum_{i=0}^s \sum_{j=0}^2 i (a_i(f)||\alpha) (g_j|\alpha - g_j) G_2^{i+j-1} \quad (3.13)$$

From (3.5), it is clear that $g_j|\alpha - g_j = 0$ for $j = 1$ and $j = 2$. It follows that:

$$D(f)||\alpha - D(f||\alpha) = -\frac{1}{8\pi i} \sum_{i=0}^s i (a_i(f)||\alpha) (g_0|\alpha - g_0) G_2^{i-1} = -\frac{1}{8\pi i} (g_0|\alpha - g_0) \cdot \left(\sum_{i=0}^s i (a_i(f)||\alpha) G_2^{i-1} \right)$$

This proves the result of (a). The result of part (b) is clear from the definition in (3.7). \square

We note here that it follows from (3.8) that for any $\alpha, \beta \in GL_2^+(\mathbb{Q})$, we have:

$$\nu_{\alpha\beta}^{(m)} = \nu_\alpha^{(m)}|\beta + \nu_\beta^{(m)} \quad \forall m \geq 1 \quad (3.14)$$

Additionally, since each G_4^m is a modular form, we know that when $\alpha \in SL_2(\mathbb{Z})$:

$$\nu_\alpha^{(m)} = -\frac{5}{24}(G_4^m|\alpha - G_4^m) = 0 \quad \forall \alpha \in SL_2(\mathbb{Z}), m \geq 1 \quad (3.15)$$

Moreover, from the definitions in (3.6) and (3.7) respectively, it is easily verified that D and $\{W_k\}_{k \geq 1}$ are derivations on the quasimodular tower \mathcal{QM} . We now proceed to define operators on the quasimodular Hecke algebra $\mathcal{Q}(\Gamma)$ for some principal congruence subgroup $\Gamma = \Gamma(N)$. Choose $F \in \mathcal{Q}(\Gamma)$. We set:

$$\begin{aligned} D, W_k, \phi^{(m)} : \mathcal{Q}(\Gamma) &\longrightarrow \mathcal{Q}(\Gamma) \quad k \geq 1, m \geq 1 \\ D(F)_\alpha &:= D(F_\alpha) \quad W_k(F)_\alpha := W_k(F_\alpha) \quad \phi^{(m)}(F)_\alpha := \nu_\alpha^{(m)} \cdot F_\alpha \quad \forall \alpha \in GL_2^+(\mathbb{Q}) \end{aligned} \quad (3.16)$$

From Lemma 3.1 and the properties of $\nu_\alpha^{(m)}$ described in (3.14) and (3.15), it may be easily verified that the operators D , W_k and $\phi^{(m)}$ in (3.16) are well defined on $\mathcal{Q}(\Gamma)$. We will now compute the commutators of the operators D , $\{W_k\}_{k \geq 1}$ and $\{\phi^{(m)}\}_{m \geq 1}$ on $\mathcal{Q}(\Gamma)$. In order to describe these commutators, we need one more operator E :

$$E : \mathcal{QM} \longrightarrow \mathcal{QM} \quad f \mapsto G_4 \cdot f \quad (3.17)$$

Since G_4 is a modular form of level $\Gamma(1) = SL_2(\mathbb{Z})$, i.e., $G_4|\gamma = G_4$ for any $\gamma \in SL_2(\mathbb{Z})$, it is clear that E induces a well defined operator on $\mathcal{Q}(\Gamma)$:

$$E : \mathcal{Q}(\Gamma) \longrightarrow \mathcal{Q}(\Gamma) \quad E(F)_\alpha := E(F_\alpha) = G_4 \cdot F_\alpha \quad \forall F \in \mathcal{Q}(\Gamma), \alpha \in GL_2^+(\mathbb{Q}) \quad (3.18)$$

We will now describe the commutator relations between the operators D , E , $\{E^l W_k\}_{k \geq 1, l \geq 0}$ and $\{\phi^{(m)}\}_{m \geq 1}$ on $\mathcal{Q}(\Gamma)$.

Proposition 3.2. *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup and let $\mathcal{Q}(\Gamma)$ be the algebra of quasimodular Hecke operators of level Γ . The operators D , E , $\{E^l W_k\}_{k \geq 1, l \geq 0}$ and $\{\phi^{(m)}\}_{m \geq 1}$ on $\mathcal{Q}(\Gamma)$ satisfy the following relations:*

$$\begin{aligned} [E, E^l W_k] &= 0 \quad [E, D] = 0 \quad [E, \phi^{(m)}] = 0 \quad [D, \phi^{(m)}] = 0 \quad [W_k, \phi^{(m)}] = 0 \quad [\phi^{(m)}, \phi^{(m')}] = 0 \\ [E^l W_k, D] &= \frac{5}{24}(k-1)(E^{l+1} W_{k-1}) - \frac{1}{2}(k-3)E^l W_{k+1} \end{aligned} \quad (3.19)$$

Proof. For any $F \in \mathcal{Q}(\Gamma)$ and any $\alpha \in GL_2^+(\mathbb{Q})$, by definition, we know that $D(F)_\alpha = D(F_\alpha)$, $W_k(F)_\alpha = W_k(F_\alpha)$ and $E(F)_\alpha = E(F_\alpha)$. Hence, in order to prove that $[E, W_k] = 0$ and $[E, D] = 0$, it suffices to show that $[E, W_k](f) = 0$ and $[E, D](f) = 0$ respectively for any element $f \in \mathcal{QM}$. Both of these are easily verified from the definitions of D and W_k in (3.6) and (3.7) respectively. Further, since $[E, W_k] = 0$, it is clear that $[E, E^l W_k] = 0$.

Similarly, in order to prove the expression for $[E^l W_k, D]$, it suffices to prove that:

$$[E^l W_k, D](f) = \frac{5}{24}(k-1)(E^{l+1} W_{k-1})(f) - \frac{1}{2}(k-3)E^l W_{k+1}(f) \quad (3.20)$$

for any $f \in \mathcal{QM}$. Further, it suffices to consider the case where $f = \sum_{i=0}^s a_i(f) G_2^i$ where the $a_i(f) \in \mathcal{M}$. We now have:

$$\begin{aligned} W_k D(f) &= -\frac{1}{8\pi i} W_k \left(\sum_{i=0}^s \sum_{j=0}^2 i a_i(f) g_j G_2^{i+j-1} \right) = -\frac{1}{8\pi i} \sum_{i=0}^s \sum_{j=0}^2 i(i+j-1) a_i(f) g_j G_2^{i+j+k-3} \\ D W_k(f) &= D \left(\sum_{i=0}^s i a_i(f) G_2^{i+k-2} \right) = -\frac{1}{8\pi i} \sum_{i=0}^s \sum_{j=0}^2 i(i+k-2) a_i(f) g_j G_2^{i+j+k-3} \end{aligned} \quad (3.21)$$

It follows from (3.21) that:

$$\begin{aligned} [W_k, D](f) &= -\frac{1}{8\pi i} \sum_{i=0}^s \sum_{j=0}^2 i j a_i(f) g_j G_2^{i+j+k-3} + \frac{1}{8\pi i} \sum_{i=0}^s \sum_{j=0}^2 i(k-1) a_i(f) g_j G_2^{i+j+k-3} \\ &= -\frac{2g_2}{8\pi i} \sum_{i=0}^s i a_i(f) G_2^{i+k-1} + (k-1) \frac{1}{8\pi i} \sum_{i=0}^s i a_i(f) g_0 G_2^{i+k-3} + (k-1) \frac{g_2}{8\pi i} \sum_{i=0}^s i a_i(f) G_2^{i+k-1} \end{aligned}$$

where the second equality uses the fact that $g_1 = 0$. Further, since $g_0 = \frac{5\pi i}{3} G_4$ and $g_2 = -4\pi i$, it follows from (3.1) that we have:

$$\begin{aligned} [W_k, D](f) &= \frac{5}{24}(k-1) \sum_{i=0}^s i G_4 a_i(f) G_2^{i+k-3} - \frac{1}{2}(k-3) \sum_{i=0}^s i a_i(f) G_2^{i+k-1} \\ &= \frac{5}{24}(k-1)(E W_{k-1})(f) - \frac{1}{2}(k-3)W_{k+1}(f) \end{aligned} \quad (3.22)$$

Finally, since E commutes with $\{W_k\}_{k \geq 1}$ and D , it follows from (3.22) that:

$$[E^l W_k, D] = \frac{5}{24}(k-1)(E^{l+1} W_{k-1}) - \frac{1}{2}(k-3)E^l W_{k+1} \quad \forall k \geq 1, l \geq 0 \quad (3.23)$$

as operators on $\mathcal{Q}(\Gamma)$. Finally, it may be easily verified from the definitions that $[E, \phi^{(m)}] = [D, \phi^{(m)}] = [W_k, \phi^{(m)}] = 0$. \square

The operators $\{E^l W_k\}_{k \geq 1, l \geq 0}$ appearing in Proposition 3.2 above can be described more succinctly as:

$$T_k^l : \mathcal{QM} \longrightarrow \mathcal{QM} \quad T_k^l := E^l W_k \quad \forall k \geq 1, l \geq 0 \quad (3.24)$$

and

$$T_k^l : \mathcal{Q}(\Gamma) \longrightarrow \mathcal{Q}(\Gamma) \quad T_k^l(F)_\alpha := T_k^l(F_\alpha) = E^l W_k(F_\alpha) \quad \forall F \in \mathcal{Q}(\Gamma), \alpha \in GL_2^+(\mathbb{Q}) \quad (3.25)$$

We are now ready to describe the Lie algebra action on $\mathcal{Q}(\Gamma)$.

Proposition 3.3. *Let \mathcal{L} be the Lie algebra generated by the symbols D , $\{T_k^l\}_{k \geq 1, l \geq 0}$, $\{\phi^{(m)}\}_{m \geq 1}$ along with the following relations between the commutators:*

$$\begin{aligned} [T_k^l, T_{k'}^{l'}] &= (k' - k)T_{k+k'}^{l+l'} \\ [D, \phi^{(m)}] &= 0 \quad [T_k^l, \phi^{(m)}] = 0 \quad [\phi^{(m)}, \phi^{(m')}] = 0 \\ [T_k^l, D] &= \frac{5}{24}(k-1)T_{k-1}^{l+1} - \frac{1}{2}(k-3)T_{k+1}^l \end{aligned} \quad (3.26)$$

Then, for any principal congruence subgroup $\Gamma = \Gamma(N)$, we have a Lie action of \mathcal{L} on the algebra of quasimodular Hecke operators $\mathcal{Q}(\Gamma)$ of level Γ .

Proof. For any $k \geq 1$ and $l \geq 0$, T_k^l has been defined to be the operator $E^l W_k$ on $\mathcal{Q}(\Gamma)$. We want to verify that:

$$[T_k^l, T_{k'}^{l'}] = (k - k')T_{k+k'}^{l+l'} \quad \forall k, k' \geq 1, l, l' \geq 0 \quad (3.27)$$

As in the proof of Proposition 3.2, it suffices to show that the relation in (3.27) holds for any $f \in \mathcal{QM}$. As before, we let $f = \sum_{i=0}^s a_i(f)G_2^i$ where each $a_i(f) \in \mathcal{M}$. We now have:

$$\begin{aligned} T_k^l T_{k'}^{l'}(f) &= T_k^l \left(\sum_{i=0}^s i a_i(f) G_4^{l'} \cdot G_2^{i+k'-2} \right) = \sum_{i=0}^s i(i+k'-2) a_i(f) G_4^{l+l'} \cdot G_2^{i+k'+k-4} \\ T_{k'}^{l'} T_k^l(f) &= T_{k'}^{l'} \left(\sum_{i=0}^s i a_i(f) G_4^l \cdot G_2^{i+k-2} \right) = \sum_{i=0}^s i(i+k-2) a_i(f) G_4^{l+l'} \cdot G_2^{i+k'+k-4} \end{aligned} \quad (3.28)$$

From (3.28) it follows that:

$$[T_k^l, T_{k'}^{l'}](f) = (k' - k) \sum_{i=0}^s i a_i(f) G_4^{l+l'} \cdot G_2^{i+k'+k-4} = (k' - k) T_{k+k'}^{l+l'}(f) \quad (3.29)$$

Hence, the relation (3.27) holds for the operators $T_k^l, T_{k'}^{l'}$ acting on $\mathcal{Q}(\Gamma)$. The remaining relations in (3.26) for the Lie action of \mathcal{L} on $\mathcal{Q}(\Gamma)$ follow from (3.19). \square

Lemma 3.4. *Let $f \in \mathcal{QM}$ be an element of the quasimodular tower and let $\alpha \in GL_2^+(\mathbb{Q})$. Then, for any $k \geq 1$, $l \geq 0$, the operator $T_k^l : \mathcal{QM} \rightarrow \mathcal{QM}$ satisfies:*

$$T_k^l(f)||\alpha = T_k^l(f)||\alpha - \frac{24}{5}\nu_\alpha^{(l)} \cdot (T_k^0(f)||\alpha) \quad (3.30)$$

Proof. For the sake of definiteness, we assume that $f = \sum_{i=0}^s a_i(f) \cdot G_2^i$ with each $a_i(f) \in \mathcal{M}$. We now compute:

$$\begin{aligned} T_k^l(f)||\alpha &= (E^l W_k)(f)||\alpha & T_k^l(f)||\alpha &= (E^l W_k)(f)||\alpha \\ &= \left(\sum_{i=0}^s i G_4^l \cdot a_i(f) G_2^{i+k-2} \right) ||\alpha & &= (E^l W_k) \left(\sum_{i=0}^s (a_i(f)||\alpha) G_2^i \right) \\ &= \sum_{i=0}^s i (G_4^l ||\alpha) \cdot (a_i(f)||\alpha) G_2^{i+k-2} & &= \sum_{i=0}^s i (G_4^l) \cdot (a_i(f)||\alpha) G_2^{i+k-2} \end{aligned} \quad (3.31)$$

Subtracting, it follows that:

$$T_k^l(f)||\alpha - T_k^l(f)||\alpha = (G_4^l ||\alpha - G_4^l) \cdot \left(\sum_{i=0}^s i (a_i(f)||\alpha) G_2^{i+k-2} \right) = -\frac{24}{5}\nu_\alpha^{(l)} \cdot (W_k(f)||\alpha) \quad (3.32)$$

Putting $T_k^0 = E^0 W_k = W_k$, we have the result. \square

Proposition 3.5. *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup and let $\mathcal{Q}(\Gamma)$ be the algebra of quasimodular Hecke operators of level Γ . Then, for any $k \geq 1$, $l \geq 0$, the operator T_k^l satisfies:*

$$T_k^l(F^1 * F^2) = T_k^l(F^1) * F^2 + F^1 * T_k^l(F^2) + \frac{24}{5}(\phi^{(l)}(F^1) * T_k^0(F^2))_\alpha \quad \forall F^1, F^2 \in \mathcal{Q}(\Gamma) \quad (3.33)$$

Further, the operators $\{T_k^l\}_{k \geq 1, l \geq 0}$ are all derivations on the algebra $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), *^r)$.

Proof. We know that $T_k^l = E^l W_k$ and that W_k is a derivation on \mathcal{QM} . We choose quasimodular Hecke operators $F^1, F^2 \in \mathcal{Q}(\Gamma)$. Then, for any $\alpha \in GL_2^+(\mathbb{Q})$, we know that:

$$\begin{aligned} T_k^l(F^1 * F^2)_\alpha &= E^l W_k \left(\sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot (F_{\alpha\beta^{-1}}^2 ||\beta) \right) \\ &= \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} E^l W_k \left(F_\beta^1 \cdot (F_{\alpha\beta^{-1}}^2 ||\beta) \right) \\ &= \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} G_4^l \cdot W_k(F_\beta^1) \cdot (F_{\alpha\beta^{-1}}^2 ||\beta) + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot G_4^l \cdot W_k(F_{\alpha\beta^{-1}}^2 ||\beta) \\ &= \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} G_4^l \cdot W_k(F_\beta^1) \cdot (F_{\alpha\beta^{-1}}^2 ||\beta) + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot G_4^l \cdot (W_k(F_{\alpha\beta^{-1}}^2 ||\beta)) \\ &= (T_k^l(F^1) * F^2)_\alpha + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot (G_4^l ||\beta) \cdot (W_k(F_{\alpha\beta^{-1}}^2 ||\beta)) - \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot (G_4^l ||\beta - G_4^l) \cdot (W_k(F_{\alpha\beta^{-1}}^2 ||\beta)) \\ &= (T_k^l(F^1) * F^2)_\alpha + (F^1 * T_k^l(F^2))_\alpha + \frac{24}{5} \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot \nu_\beta^{(l)} \cdot (W_k(F_{\alpha\beta^{-1}}^2 ||\beta)) \\ &= (T_k^l(F^1) * F^2)_\alpha + (F^1 * T_k^l(F^2))_\alpha + \frac{24}{5}(\phi^{(l)}(F^1) * T_k^0(F^2))_\alpha \end{aligned}$$

where it is understood that $\phi^{(0)} = 0$. This proves (3.33). Further, since $\nu_\beta^{(l)} = 0$ for any $\beta \in SL_2(\mathbb{Z})$, when we consider the product $*^r$ defined in (2.19) on the algebra $\mathcal{Q}^r(\Gamma)$, the calculation above reduces to

$$T_k^l(F^1 *^r F^2) = T_k^l(F^1) *^r F^2 + F^1 *^r T_k^l(F^2) \quad (3.34)$$

Hence, each T_k^l is a derivation on $\mathcal{Q}^r(\Gamma)$. \square

Proposition 3.6. Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup and let $\mathcal{Q}(\Gamma)$ be the algebra of quasimodular Hecke operators of level Γ .

(a) The operator $D : \mathcal{Q}(\Gamma) \rightarrow \mathcal{Q}(\Gamma)$ on the algebra $(\mathcal{Q}(\Gamma), *)$ satisfies:

$$D(F^1 * F^2) = D(F^1) * F^2 + F^1 * D(F^2) - \phi^{(1)}(F^1) * T_1^0(F^2) \quad \forall F^1, F^2 \in \mathcal{Q}(\Gamma) \quad (3.35)$$

When we consider the product $*^r$, the operator D becomes a derivation on the algebra $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), *^r)$, i.e.:

$$D(F^1 *^r F^2) = D(F^1) *^r F^2 + F^1 *^r D(F^2) \quad \forall F^1, F^2 \in \mathcal{Q}^r(\Gamma) \quad (3.36)$$

(b) The operators $\{W_k\}_{k \geq 1}$ and $\{\phi^{(m)}\}_{m \geq 1}$ are derivations on $\mathcal{Q}(\Gamma)$, i.e.,

$$\begin{aligned} W_k(F^1 * F^2) &= W_k(F^1) * F^2 + F^1 * W_k(F^2) \\ \phi^{(m)}(F^1 * F^2) &= \phi^{(m)}(F^1) * F^2 + F^1 * \phi^{(m)}(F^2) \end{aligned} \quad (3.37)$$

for any $F^1, F^2 \in \mathcal{Q}(\Gamma)$. Additionally, $\{\phi^{(m)}\}_{m \geq 1}$ and $\{W_k\}_{k \geq 1}$ are also derivations on the algebra $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), *^r)$.

Proof. (a) We choose quasimodular Hecke operators $F^1, F^2 \in \mathcal{Q}(\Gamma)$. We have mentioned before that D is a derivation on \mathcal{QM} . Then, for any $\alpha \in GL_2^+(\mathbb{Q})$, we have:

$$\begin{aligned} D(F^1 * F^2)_\alpha &= D \left(\sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot (F_{\alpha\beta^{-1}}^2 || \beta) \right) \\ &= \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} D \left(F_\beta^1 \cdot (F_{\alpha\beta^{-1}}^2 || \beta) \right) \\ &= \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} D(F_\beta^1) \cdot (F_{\alpha\beta^{-1}}^2 || \beta) + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot D(F_{\alpha\beta^{-1}}^2 || \beta) \\ &= (D(F^1) * F^2)_\alpha + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot (D(F_{\alpha\beta^{-1}}^2) || \beta) - \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot \nu_\beta^{(1)} \cdot (W_1(F_{\alpha\beta^{-1}}^2) || \beta) \\ &= (D(F^1) * F^2)_\alpha + (F^1 * D(F^2))_\alpha - (\phi^{(1)}(F^1) * T_1^0(F^2))_\alpha \end{aligned}$$

This proves (3.35). In order to prove (3.36), we note that $\nu_\beta^{(1)} = 0$ for any $\beta \in SL_2(\mathbb{Z})$ (see (3.15)). Hence, when we use the product $*^r$ defined in (2.19), the calculation above reduces to

$$D(F^1 *^r F^2) = D(F^1) *^r F^2 + F^1 *^r D(F^2) \quad (3.38)$$

for any $F^1, F^2 \in \mathcal{Q}^r(\Gamma)$.

(b) For any $F^1, F^2 \in \mathcal{Q}(\Gamma)$ and knowing from (3.14) that $\nu_\alpha^{(m)} = \nu_\beta^{(m)} + \nu_{\alpha\beta^{-1}}^{(m)} || \beta$, we have:

$$\begin{aligned} \phi^{(m)}(F^1 * F^2)_\alpha &= \nu_\alpha^{(m)} \cdot \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot (F_{\alpha\beta^{-1}}^2 || \beta) \\ &= \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} (\nu_\beta^{(m)} \cdot F_\beta^1) \cdot (F_{\alpha\beta^{-1}}^2 || \beta) + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot (\nu_{\alpha\beta^{-1}}^{(m)} || \beta) \cdot (F_{\alpha\beta^{-1}}^2 || \beta) \\ &= \phi^{(m)}(F^1) * F^2 + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot ((\nu_{\alpha\beta^{-1}}^{(m)} \cdot F_{\alpha\beta^{-1}}^2) || \beta) \\ &= \phi^{(m)}(F^1) * F^2 + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot (\phi^{(m)}(F^2)_{\alpha\beta^{-1}} || \beta) \\ &= \phi^{(m)}(F^1) * F^2 + F^1 * \phi^{(m)}(F^2) \end{aligned} \quad (3.39)$$

The fact that each W_k is also a derivation on $\mathcal{Q}(\Gamma)$ now follows from a similar calculation using the fact that W_k is a derivation on the quasimodular tower \mathcal{QM} and that $W_k(f)|\alpha = W_k(f|\alpha)$ for any $f \in \mathcal{QM}$, $\alpha \in GL_2^+(\mathbb{Q})$ (from (3.11)). Finally, a similar calculation may be used to verify that $\{W_k\}_{k \geq 1}$ and $\{\phi^{(m)}\}_{m \geq 1}$ are all derivations on $\mathcal{Q}^r(\Gamma)$. \square

We now introduce the Hopf algebra \mathcal{H} that acts on $\mathcal{Q}^r(\Gamma)$. The Hopf algebra \mathcal{H} is the universal enveloping algebra $\mathcal{U}(\mathcal{L})$ of the Lie algebra \mathcal{L} defined by generators D , $\{T_k^l\}_{k \geq 1, l \geq 0}$, $\{\phi^{(m)}\}_{m \geq 1}$ satisfying the following relations:

$$\begin{aligned} [T_k^l, T_{k'}^{l'}] &= (k' - k)T_{k+k'-2}^{l+l'} & [T_k^l, D] &= \frac{5}{24}(k-1)T_{k-1}^{l+1} - \frac{1}{2}(k-3)T_{k+1}^l \\ [D, \phi^{(m)}] &= 0 & [T_k^l, \phi^{(m)}] &= 0 & [\phi^{(m)}, \phi^{(m')}] &= 0 \end{aligned} \quad (3.40)$$

As such, the coproduct $\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ is defined by:

$$\Delta(D) = D \otimes 1 + 1 \otimes D \quad \Delta(T_k^l) = T_k^l \otimes 1 + 1 \otimes T_k^l \quad \Delta(\phi^{(m)}) = \phi^{(m)} \otimes 1 + 1 \otimes \phi^{(m)} \quad (3.41)$$

We will now show that \mathcal{H} has a Hopf action on the algebra $\mathcal{Q}^r(\Gamma)$.

Proposition 3.7. *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. Then, there is a Hopf action of \mathcal{H} on the algebra $\mathcal{Q}^r(\Gamma)$, i.e.,*

$$h(F^1 *^r F^2) = \sum h_{(1)}(F^1) *^r h_{(2)}(F^2) \quad \forall F^1, F^2 \in \mathcal{Q}^r(\Gamma), h \in \mathcal{H} \quad (3.42)$$

where $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for any $h \in \mathcal{H}$.

Proof. In order to prove (3.42), it suffices to verify the relation for D and each of $\{T_k^l\}_{k \geq 1, l \geq 0}$, $\{\phi^{(m)}\}_{m \geq 1}$. From Proposition 3.5 and Proposition 3.6, we know that for $F^1, F^2 \in \mathcal{Q}^r(\Gamma)$ and any $k \geq 1, l \geq 0, m \geq 1$:

$$\begin{aligned} D(F^1 *^r F^2) &= D(F^1) *^r F^2 + F^1 *^r D(F^2) \\ T_k^l(F^1 *^r F^2) &= T_k^l(F^1) *^r F^2 + F^1 *^r T_k^l(F^2) \\ \phi^{(m)}(F^1 *^r F^2) &= \phi^{(m)}(F^1) *^r F^2 + F^1 *^r \phi^{(m)}(F^2) \end{aligned} \quad (3.43)$$

Comparing with the expressions for the coproduct in (3.41), it is clear that (3.42) holds for each $h \in \mathcal{H}$. \square

3.2 The operators X , Y and $\{\delta_n\}$ of Connes and Moscovici

Let $\Gamma = \Gamma(N)$ be a congruence subgroup. In this subsection, we will show that the algebra $\mathcal{Q}(\Gamma)$ carries an action of the Hopf algebra \mathcal{H}_1 of Connes and Moscovici [2]. The Hopf algebra \mathcal{H}_1 is part of a larger family $\{\mathcal{H}_n\}_{n \geq 1}$ of Hopf algebras defined in [2] and \mathcal{H}_1 is the Hopf algebra corresponding to ‘‘codimension 1 foliations’’. As an algebra, \mathcal{H}_1 is identical to the universal enveloping algebra $\mathcal{U}(\mathcal{L}_1)$ of the Lie algebra \mathcal{L}_1 generated by X , Y , $\{\delta_n\}_{n \geq 1}$ satisfying the commutator relations:

$$[Y, X] = X \quad [X, \delta_n] = \delta_{n+1} \quad [Y, \delta_n] = n\delta_n \quad [\delta_k, \delta_l] = 0 \quad \forall k, l, n \geq 1 \quad (3.44)$$

Further, the coproduct $\Delta : \mathcal{H}_1 \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_1$ on \mathcal{H}_1 is determined by:

$$\begin{aligned} \Delta(X) &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y \\ \Delta(Y) &= Y \otimes 1 + 1 \otimes Y & \Delta(\delta_1) &= \delta_1 \otimes 1 + 1 \otimes \delta_1 \end{aligned} \quad (3.45)$$

Finally, the antipode $S : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is given by:

$$S(X) = -X + \delta_1 Y \quad S(Y) = -Y \quad S(\delta_1) = -\delta_1 \quad (3.46)$$

Following Connes and Moscovici [3], we define the operators X and Y on the modular tower: for any congruence subgroup $\Gamma = \Gamma(N)$, we set:

$$Y : \mathcal{M}_k(\Gamma) \longrightarrow \mathcal{M}_k(\Gamma) \quad Y(f) := \frac{k}{2} f \quad \forall f \in \mathcal{M}_k(\Gamma) \quad (3.47)$$

Further, the operator $X : \mathcal{M}_k(\Gamma) \longrightarrow \mathcal{M}_{k+2}(\Gamma)$ is the Ramanujan differential operator on modular forms:

$$X(f) := \frac{1}{2\pi i} \frac{d}{dz}(f) - \frac{1}{12\pi i} \frac{d}{dz}(\log \Delta) \cdot Y(f) \quad \forall f \in \mathcal{M}_k(\Gamma) \quad (3.48)$$

where $\Delta(z)$ is the well known modular form of weight 12 given by:

$$\Delta(z) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q = e^{2\pi i z} \quad (3.49)$$

We start by extending these operators to the quasimodular tower \mathcal{QM} . Let $f \in \mathcal{QM}_k^s(\Gamma)$ be a quasimodular form. Then, we can express $f = \sum_{i=0}^s a_i(f) G_2^i$ where $a_i(f) \in \mathcal{M}_{k-2i}(\Gamma)$. We set:

$$X(f) = \sum_{i=0}^s X(a_i(f)) \cdot G_2^i \quad Y(f) = \sum_{i=0}^s Y(a_i(f)) \cdot G_2^i \quad (3.50)$$

From (3.50), it is clear that X and Y are derivations on \mathcal{QM} .

Lemma 3.8. *Let $f \in \mathcal{QM}$ be an element of the quasimodular tower. Then, for any $\alpha \in GL_2^+(\mathbb{Q})$, we have:*

$$X(f)||\alpha = X(f||\alpha) + (\mu_{\alpha^{-1}} \cdot Y(f))||\alpha \quad (3.51)$$

where, for any $\delta \in GL_2^+(\mathbb{Q})$, we set:

$$\mu_\delta := \frac{1}{12\pi i} \frac{d}{dz} \log \frac{\Delta|\delta}{\Delta} \quad (3.52)$$

Further, we have $Y(f||\alpha) = Y(f)||\alpha$.

Proof. Following [3, Lemma 5], we know that for any $g \in \mathcal{M}$, we have:

$$X(g)|\alpha = X(g|\alpha) + (\mu_{\alpha^{-1}} \cdot Y(g))|\alpha \quad \forall \alpha \in GL_2^+(\mathbb{Q}) \quad (3.53)$$

It suffices to consider the case $f \in \mathcal{QM}_k^s(\Gamma)$ for some congruence subgroup Γ . If we express $f \in \mathcal{QM}_k^s(\Gamma)$ as $f = \sum_{i=0}^s a_i(f) G_2^i$ with $a_i(f) \in \mathcal{M}_{k-2i}(\Gamma)$, it follows that:

$$X(a_i(f))|\alpha = X(a_i(f)|\alpha) + (\mu_{\alpha^{-1}} \cdot Y(a_i(f)))|\alpha \quad \forall \alpha \in GL_2^+(\mathbb{Q}) \quad (3.54)$$

for each $0 \leq i \leq s$. Combining (3.54) with the definitions of X and Y on the quasimodular tower in (3.50), we can easily prove (3.51). Finally, it is clear from the definition of Y that $Y(f||\alpha) = Y(f)||\alpha$. \square

From the definition of μ_δ in (3.52), one may verify that (see [3, § 3]):

$$\mu_{\delta_1 \delta_2} = \mu_{\delta_1} | \delta_2 + \mu_{\delta_2} \quad \forall \delta_1, \delta_2 \in GL_2^+(\mathbb{Q}) \quad (3.55)$$

and that $\mu_\delta = 0$ for any $\delta \in SL_2(\mathbb{Z})$. We now define operators X, Y and $\{\delta_n\}_{n \geq 1}$ on the quasimodular Hecke algebra $\mathcal{Q}(\Gamma)$ for some congruence subgroup $\Gamma = \Gamma(N)$. Let $F \in \mathcal{Q}(\Gamma)$ be a quasimodular Hecke operator of level Γ ; then we define operators:

$$\begin{aligned} X, Y, \delta_n : \mathcal{Q}(\Gamma) &\longrightarrow \mathcal{Q}(\Gamma) \\ X(F)_\alpha &:= X(F_\alpha) \quad Y(F)_\alpha := Y(F_\alpha) \quad \delta_n(F)_\alpha = X^{n-1}(\mu_\alpha) \cdot F_\alpha \quad \forall \alpha \in GL_2^+(\mathbb{Q}) \end{aligned} \quad (3.56)$$

We will now show that the Lie algebra \mathcal{L}_1 with generators $X, Y, \{\delta_n\}_{n \geq 1}$ satisfying the commutator relations in (3.44) acts on the algebra $\mathcal{Q}(\Gamma)$. Additionally, in order to give a Lie action on the algebra $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), *^r)$, we define at this juncture the smaller Lie algebra $\mathfrak{l}_1 \subseteq \mathcal{L}_1$ with generators X and Y satisfying the relation

$$[Y, X] = X \quad (3.57)$$

Further, we consider the Hopf algebra \mathfrak{h}_1 that arises as the universal enveloping algebra $\mathcal{U}(\mathfrak{l}_1)$ of the Lie algebra \mathfrak{l}_1 . We will show that \mathcal{H}_1 (resp. \mathfrak{h}_1) has a Hopf action on the algebra $\mathcal{Q}(\Gamma)$ (resp. $\mathcal{Q}^r(\Gamma)$). We start by describing the Lie actions.

Proposition 3.9. *Let \mathcal{L}_1 be the Lie algebra with generators X, Y and $\{\delta_n\}_{n \geq 1}$ satisfying the following commutator relations:*

$$[Y, X] = X \quad [X, \delta_n] = \delta_{n+1} \quad [Y, \delta_n] = n\delta_n \quad [\delta_k, \delta_l] = 0 \quad \forall k, l, n \geq 1 \quad (3.58)$$

Then, for any given congruence subgroup $\Gamma = \Gamma(N)$ of $SL_2(\mathbb{Z})$, we have a Lie action of \mathcal{L}_1 on the module $\mathcal{Q}(\Gamma)$.

Proof. From [3, § 3], we know that for any element $g \in \mathcal{M}$ of the modular tower, we have $[Y, X](g) = X(g)$. Since the action of X and Y on the quasimodular tower \mathcal{QM} (see (3.50)) is naturally extended from their action on \mathcal{M} , it follows that $[Y, X] = X$ on the quasimodular tower \mathcal{QM} . In particular, given any quasimodular Hecke operator $F \in \mathcal{Q}(\Gamma)$ and any $\alpha \in GL_2^+(\mathbb{Q})$, we have $[Y, X](F_\alpha) = X(F_\alpha)$ for the element $F_\alpha \in \mathcal{QM}$. By definition, $X(F)_\alpha = X(F_\alpha)$ and $Y(F)_\alpha = Y(F)_\alpha$ and hence $[Y, X] = X$ holds for the action of X and Y on $\mathcal{Q}(\Gamma)$.

Further, since X is a derivation on \mathcal{QM} and $\delta_n(F)_\alpha = X^{n-1}(\mu_\alpha) \cdot F_\alpha$, we have

$$\begin{aligned} [X, \delta_n](F)_\alpha &= X(X^{n-1}(\mu_\alpha) \cdot F_\alpha) - X^{n-1}(\mu_\alpha) \cdot X(F_\alpha) \\ &= X(X^{n-1}(\mu_\alpha)) \cdot F_\alpha = X^n(\mu_\alpha) \cdot F_\alpha = \delta_{n+1}(F)_\alpha \end{aligned} \quad (3.59)$$

Similarly, since $\mu_\alpha \in \mathcal{M} \subseteq \mathcal{QM}$ is of weight 2 and Y is a derivation on \mathcal{QM} , we have:

$$\begin{aligned} [Y, \delta_n](F)_\alpha &= Y(X^{n-1}(\mu_\alpha) \cdot F_\alpha) - X^{n-1}(\mu_\alpha) \cdot Y(F_\alpha) \\ &= Y(X^{n-1}(\mu_\alpha)) \cdot F_\alpha = nX^{n-1}(\mu_\alpha) \cdot F_\alpha = n\delta_n(F)_\alpha \end{aligned} \quad (3.60)$$

Finally, we can verify easily that $[\delta_k, \delta_l] = 0$ for any $k, l \geq 1$. \square

From Proposition 3.9, it is also clear that the smaller Lie algebra $\mathfrak{l}_1 \subseteq \mathcal{L}_1$ has a Lie action on the module $\mathcal{Q}(\Gamma)$.

Lemma 3.10. *Let $\Gamma = \Gamma(N)$ be a congruence subgroup of $SL_2(\mathbb{Z})$ and let $\mathcal{Q}(\Gamma)$ be the algebra of quasimodular Hecke operators of level Γ . Then, the operator $X : \mathcal{Q}(\Gamma) \rightarrow \mathcal{Q}(\Gamma)$ on the algebra $(\mathcal{Q}(\Gamma), *)$ satisfies:*

$$X(F^1 * F^2) = X(F^1) * F^2 + F^1 * X(F^2) + \delta_1(F^1) * Y(F^2) \quad \forall F^1, F^2 \in \mathcal{Q}(\Gamma) \quad (3.61)$$

When we consider the product $*^r$, the operator X becomes a derivation on the algebra $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), *^r)$, i.e.:

$$X(F^1 *^r F^2) = X(F^1) *^r F^2 + F^1 *^r X(F^2) \quad \forall F^1, F^2 \in \mathcal{Q}^r(\Gamma) \quad (3.62)$$

Proof. We choose quasimodular Hecke operators $F^1, F^2 \in \mathcal{Q}(\Gamma)$. Using (3.55), we also note that

$$0 = \mu_1 = \mu_{\beta^{-1}}|\beta + \mu_\beta \quad \forall \beta \in GL_2^+(\mathbb{Q}) \quad (3.63)$$

We have mentioned before that X is a derivation on \mathcal{QM} . Then, for any $\alpha \in GL_2^+(\mathbb{Q})$, we have:

$$\begin{aligned} X(F^1 * F^2)_\alpha &= X \left(\sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot (F_{\alpha\beta^{-1}}^2 || \beta) \right) \\ &= \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} X \left(F_\beta^1 \cdot (F_{\alpha\beta^{-1}}^2 || \beta) \right) \\ &= \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} X(F_\beta^1) \cdot (F_{\alpha\beta^{-1}}^2 || \beta) + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot X(F_{\alpha\beta^{-1}}^2 || \beta) \\ &= (X(F^1) * F^2)_\alpha + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot (X(F_{\alpha\beta^{-1}}^2 || \beta) - \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot ((\mu_{\beta^{-1}} \cdot Y(F_{\alpha\beta^{-1}}^2)) || \beta)) \\ &= (X(F^1) * F^2)_\alpha + (F^1 * X(F^2))_\alpha + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} (F_\beta^1 \cdot \mu_\beta) \cdot (Y(F_{\alpha\beta^{-1}}^2) || \beta) \\ &= (X(F^1) * F^2)_\alpha + (F^1 * X(F^2))_\alpha + (\delta_1(F^1) * Y(F^2))_\alpha \end{aligned}$$

This proves (3.61). In order to prove (3.62), we note that $\mu_\beta = 0$ for any $\beta \in SL_2(\mathbb{Z})$. Hence, if we use the product $*^r$, the calculation above reduces to

$$X(F^1 *^r F^2) = X(F^1) *^r F^2 + F^1 *^r X(F^2) \quad (3.64)$$

for any $F^1, F^2 \in \mathcal{Q}^r(\Gamma)$. \square

Finally, we describe the Hopf action of \mathcal{H}_1 on the algebra $(\mathcal{Q}(\Gamma), *)$ as well as the Hopf action of \mathfrak{h}_1 on the algebra $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), *^r)$.

Proposition 3.11. *Let $\Gamma = \Gamma(N)$ be a congruence subgroup of $SL_2(\mathbb{Z})$. Then, the Hopf algebra \mathcal{H}_1 has a Hopf action on the quasimodular Hecke algebra $(\mathcal{Q}(\Gamma), *)$; in other words, we have:*

$$h(F^1 * F^2) = \sum h_{(1)}(F^1) \otimes h_{(2)}(F^2) \quad \forall h \in \mathcal{H}_1, F^1, F^2 \in \mathcal{Q}(\Gamma) \quad (3.65)$$

where the coproduct $\Delta : \mathcal{H}_1 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_1$ is given by $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for any $h \in \mathcal{H}_1$. Similarly, there exists a Hopf action of the Hopf algebra \mathfrak{h}_1 on the algebra $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), *^r)$.

Proof. In order to prove (3.65), it suffices to check the relation for X, Y and $\delta_1 \in \mathcal{H}_1$. For the element $X \in \mathcal{H}_1$, this is already the result of Lemma 3.10. Now, for any $F^1, F^2 \in \mathcal{Q}(\Gamma)$ and $\alpha \in GL_2^+(\mathbb{Q})$, we have:

$$\begin{aligned} \delta_1(F^1 * F^2)_\alpha &= \mu_\alpha \cdot \left(\sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot (F_{\alpha\beta^{-1}}^2 || \beta) \right) \\ &= \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} (\mu_\beta \cdot F_\beta^1) \cdot (F_{\alpha\beta^{-1}}^2 || \beta) + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot ((\mu_{\alpha\beta^{-1}} \cdot F_{\alpha\beta^{-1}}^2) || \beta) \\ &= (\delta_1(F^1) * F^2)_\alpha + (F^1 * \delta_1(F^2))_\alpha \end{aligned} \quad (3.66)$$

Further, using the fact that Y is a derivation on \mathcal{QM} and $Y(f||\alpha) = Y(f)||\alpha$ for any $f \in \mathcal{QM}$, $\alpha \in GL_2^+(\mathbb{Q})$, we can easily verify the relation (3.65) for the element $Y \in \mathcal{H}_1$. This proves (3.65) for all $h \in \mathcal{H}_1$.

Finally, in order to demonstrate the Hopf action of \mathfrak{h}_1 on $\mathcal{Q}^r(\Gamma)$, we need to check that:

$$X(F^1 *^r F^2) = X(F^1) *^r F^2 + F^1 *^r X(F^2) \quad Y(F^1 *^r F^2) = Y(F^1) *^r F^2 + F^1 *^r Y(F^2) \quad (3.67)$$

for any $F^1, F^2 \in \mathcal{Q}^r(\Gamma)$. The relation for X has already been proved in (3.64). The relation for Y is again an easy consequence of the fact that Y is a derivation on \mathcal{QM} and $Y(f||\alpha) = Y(f)||\alpha$ for any $f \in \mathcal{QM}$, $\alpha \in GL_2^+(\mathbb{Q})$. \square

4 Twisted Quasimodular Hecke operators

Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. For any $\sigma \in SL_2(\mathbb{Z})$, we have developed the theory of σ -twisted modular Hecke operators in [1]. In this section, we introduce and study the collection $\mathcal{Q}_\sigma(\Gamma)$ of quasimodular Hecke operators of level Γ twisted by σ . When $\sigma = 1$, $\mathcal{Q}_\sigma(\Gamma)$ coincides with the algebra $\mathcal{Q}(\Gamma)$ of quasimodular Hecke operators. In general, we will show that $\mathcal{Q}_\sigma(\Gamma)$ is a right $\mathcal{Q}(\Gamma)$ -module and carries a pairing:

$$(_, _) : \mathcal{Q}_\sigma(\Gamma) \otimes \mathcal{Q}_\sigma(\Gamma) \longrightarrow \mathcal{Q}_\sigma(\Gamma) \quad (4.1)$$

We recall from Section 3 the Lie algebra \mathfrak{l}_1 with two generators Y, X satisfying $[Y, X] = X$. If we let \mathfrak{h}_1 be the Hopf algebra that is the universal enveloping algebra of \mathfrak{l}_1 , we show in Section 4.1 that the pairing in (4.1) on $\mathcal{Q}_\sigma(\Gamma)$ carries a ‘‘Hopf action’’ of \mathfrak{h}_1 . In other words, we have:

$$h(F^1, F^2) = \sum (h_{(1)}(F^1), h_{(2)}(F^2)) \quad \forall h \in \mathfrak{h}_1, F^1, F^2 \in \mathcal{Q}_\sigma(\Gamma) \quad (4.2)$$

where the coproduct $\Delta : \mathfrak{h}_1 \longrightarrow \mathfrak{h}_1 \otimes \mathfrak{h}_1$ is given by $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for any $h \in \mathfrak{h}_1$. In Section 4.2, we consider operators $X_\tau : \mathcal{Q}_\sigma(\Gamma) \longrightarrow \mathcal{Q}_{\tau\sigma}(\Gamma)$ for any $\tau, \sigma \in SL_2(\mathbb{Z})$. In particular, we consider operators acting between the levels of the graded module:

$$\mathcal{Q}_\sigma(\Gamma) = \bigoplus_{m \in \mathbb{Z}} \mathcal{Q}_{\sigma(m)}(\Gamma) \quad (4.3)$$

where for any $\sigma \in SL_2(\mathbb{Z})$, we set $\sigma(m) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot \sigma$. Further, we generalize the pairing on $\mathcal{Q}_\sigma(\Gamma)$ in (4.1) to a pairing:

$$(_, _) : \mathcal{Q}_{\sigma(m)}(\Gamma) \otimes \mathcal{Q}_{\sigma(n)}(\Gamma) \longrightarrow \mathcal{Q}_{\sigma(m+n)}(\Gamma) \quad \forall m, n \in \mathbb{Z} \quad (4.4)$$

We show that the pairing in (4.4) is a special case of a more general pairing

$$(_, _) : \mathcal{Q}_{\tau_1\sigma}(\Gamma) \otimes \mathcal{Q}_{\tau_2\sigma}(\Gamma) \longrightarrow \mathcal{Q}_{\tau_1\tau_2\sigma}(\Gamma) \quad (4.5)$$

where τ_1, τ_2 are commuting matrices in $SL_2(\mathbb{Z})$. From (4.4), it is clear that we have a graded pairing on $\mathcal{Q}_\sigma(\Gamma)$ that extends the pairing on $\mathcal{Q}_\sigma(\Gamma)$. Finally, we consider the Lie algebra $\mathfrak{l}_\mathbb{Z}$ with generators $\{Z, X_n | n \in \mathbb{Z}\}$ satisfying the commutator relations:

$$[Z, X_n] = (n+1)X_n \quad [X_n, X_{n'}] = 0 \quad \forall n, n' \in \mathbb{Z} \quad (4.6)$$

Then, for $n = 0$, we have $[Z, X_0] = X_0$ and hence the Lie algebra $\mathfrak{l}_{\mathbb{Z}}$ contains the Lie algebra \mathfrak{l}_1 acting on $\mathcal{Q}_{\sigma}(\Gamma)$. Then, if we let $\mathfrak{h}_{\mathbb{Z}}$ be the Hopf algebra that is the universal enveloping algebra of $\mathfrak{l}_{\mathbb{Z}}$, we show that $\mathfrak{h}_{\mathbb{Z}}$ has a Hopf action on the pairing on $\mathcal{Q}_{\sigma}(\Gamma)$. In other words, for any $F^1, F^2 \in \mathcal{Q}_{\sigma}(\Gamma)$, we have

$$h(F^1, F^2) = \sum (h_{(1)}(F^1), h_{(2)}(F^2)) \quad \forall h \in \mathfrak{h}_{\mathbb{Z}} \quad (4.7)$$

where the coproduct $\Delta : \mathfrak{h}_{\mathbb{Z}} \longrightarrow \mathfrak{h}_{\mathbb{Z}} \otimes \mathfrak{h}_{\mathbb{Z}}$ is defined by setting $\Delta(h) := \sum h_{(1)} \otimes h_{(2)}$ for each $h \in \mathfrak{h}_{\mathbb{Z}}$.

4.1 The pairing on $\mathcal{Q}_{\sigma}(\Gamma)$ and Hopf action

Let $\sigma \in SL_2(\mathbb{Z})$ and let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. We start by defining the collection $\mathcal{Q}_{\sigma}(\Gamma)$ of quasimodular Hecke operators of level Γ twisted by σ . When $\sigma = 1$, this reduces to the definition of $\mathcal{Q}(\Gamma)$.

Definition 4.1. Choose $\sigma \in SL_2(\mathbb{Z})$ and let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. A σ -twisted quasimodular Hecke operator F of level Γ is a function of finite support:

$$F : \Gamma \backslash GL_2^+(\mathbb{Q}) \longrightarrow \mathcal{QM} \quad \Gamma\alpha \mapsto F_{\alpha} \in \mathcal{QM} \quad (4.8)$$

such that:

$$F_{\alpha\gamma} = F_{\alpha} || \sigma\gamma\sigma^{-1} \quad \forall \gamma \in \Gamma \quad (4.9)$$

We denote by $\mathcal{Q}_{\sigma}(\Gamma)$ the collection of σ -twisted quasimodular Hecke operators of level Γ .

Proposition 4.2. Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and choose some $\sigma \in SL_2(\mathbb{Z})$. Then there exists a pairing:

$$(_, _) : \mathcal{Q}_{\sigma}(\Gamma) \otimes \mathcal{Q}_{\sigma}(\Gamma) \longrightarrow \mathcal{Q}_{\sigma}(\Gamma) \quad (4.10)$$

defined as follows:

$$(F^1, F^2)_{\alpha} := \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} F_{\beta\sigma}^1 \cdot (F_{\alpha\sigma^{-1}\beta^{-1}}^2 || \sigma\beta) \quad \forall F^1, F^2 \in \mathcal{Q}_{\sigma}(\Gamma), \alpha \in GL_2^+(\mathbb{Q}) \quad (4.11)$$

Proof. We choose $\gamma \in \Gamma$. Then, for any $\beta \in SL_2(\mathbb{Z})$, we have:

$$F_{\gamma\beta\sigma}^1 = F_{\beta\sigma}^1 \quad F_{\alpha\sigma^{-1}\beta^{-1}\gamma^{-1}}^2 || \sigma\gamma\beta = F_{\alpha\sigma^{-1}\beta^{-1}}^2 || \sigma\gamma^{-1}\sigma^{-1}\sigma\gamma\beta = F_{\alpha\sigma^{-1}\beta^{-1}}^2 || \sigma\beta \quad (4.12)$$

and hence the sum in (4.11) is well defined, i.e., it does not depend on the choice of coset representatives. We have to show that $(F^1, F^2) \in \mathcal{Q}_{\sigma}(\Gamma)$. For this, we first note that $F_{\gamma\alpha\sigma^{-1}\beta^{-1}}^2 = F_{\alpha\sigma^{-1}\beta^{-1}}^2$ for any $\gamma \in \Gamma$ and hence from the expression in (4.11), it follows that $(F^1, F^2)_{\gamma\alpha} = (F^1, F^2)_{\alpha}$. On the other hand, for any $\gamma \in \Gamma$, we can write:

$$(F^1, F^2)_{\alpha\gamma} = \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} F_{\beta\sigma}^1 \cdot (F_{\alpha\gamma\sigma^{-1}\beta^{-1}}^2 || \sigma\beta) \quad (4.13)$$

We put $\delta = \beta\sigma\gamma^{-1}\sigma^{-1}$. It is clear that as β runs through all the coset representatives of Γ in $SL_2(\mathbb{Z})$, so does δ . From (4.9), we know that $F_{\delta\sigma\gamma}^1 = F_{\delta\sigma}^1 || \sigma\gamma\sigma^{-1}$. Then, we can rewrite (4.13) as:

$$\begin{aligned}
(F^1, F^2)_{\alpha\gamma} &= \sum_{\delta \in \Gamma \backslash SL_2(\mathbb{Z})} F_{\delta\sigma\gamma}^1 \cdot (F_{\alpha\sigma^{-1}\delta^{-1}}^2 || \sigma\delta\sigma\gamma\sigma^{-1}) \\
&= \sum_{\delta \in \Gamma \backslash SL_2(\mathbb{Z})} (F_{\delta\sigma}^1 || \sigma\gamma\sigma^{-1}) \cdot ((F_{\alpha\sigma^{-1}\delta^{-1}}^2 || \sigma\delta) || \sigma\gamma\sigma^{-1}) \\
&= \left(\sum_{\delta \in \Gamma \backslash SL_2(\mathbb{Z})} F_{\delta\sigma}^1 \cdot (F_{\alpha\sigma^{-1}\delta^{-1}}^2 || \sigma\delta) \right) || (\sigma\gamma\sigma^{-1}) \\
&= (F^1, F^2)_{\alpha} || \sigma\gamma\sigma^{-1}
\end{aligned} \tag{4.14}$$

It follows that $(F^1, F^2) \in \mathcal{Q}_{\sigma}(\Gamma)$ and hence we have a well defined pairing $(_, _) : \mathcal{Q}_{\sigma}(\Gamma) \otimes \mathcal{Q}_{\sigma}(\Gamma) \longrightarrow \mathcal{Q}_{\sigma}(\Gamma)$. \square

We now consider the Hopf algebra \mathfrak{h}_1 defined in Section 3.2. By definition, \mathfrak{h}_1 is the universal enveloping algebra of the Lie algebra \mathfrak{l}_1 with two generators X and Y satisfying $[Y, X] = X$. We will now show that \mathfrak{l}_1 has a Lie action on $\mathcal{Q}_{\sigma}(\Gamma)$ and that \mathfrak{h}_1 has a ‘‘Hopf action’’ with respect to the pairing on $\mathcal{Q}_{\sigma}(\Gamma)$.

Proposition 4.3. *Let $\sigma \in SL_2(\mathbb{Z})$ and let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$.*

(a) *The Lie algebra \mathfrak{l}_1 has a Lie action on $\mathcal{Q}_{\sigma}(\Gamma)$ defined by:*

$$X(F)_{\alpha} := X(F_{\alpha}) \quad Y(F)_{\alpha} := Y(F_{\alpha}) \quad \forall F \in \mathcal{Q}_{\sigma}(\Gamma), \alpha \in GL_2^+(\mathbb{Q}) \tag{4.15}$$

(b) *The universal enveloping algebra \mathfrak{h}_1 of the Lie algebra \mathfrak{l}_1 has a ‘‘Hopf action’’ with respect to the pairing on $\mathcal{Q}_{\sigma}(\Gamma)$; in other words, we have:*

$$h(F^1, F^2) = \sum (h_{(1)}(F^1), h_{(2)}(F^2)) \quad \forall F^1, F^2 \in \mathcal{Q}_{\sigma}(\Gamma), h \in \mathfrak{h}_1 \tag{4.16}$$

where the coproduct $\Delta : \mathfrak{h}_1 \longrightarrow \mathfrak{h}_1 \otimes \mathfrak{h}_1$ is given by $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for any $h \in \mathfrak{h}_1$.

Proof. (a) We need to verify that for any $F \in \mathcal{Q}_{\sigma}(\Gamma)$ and any $\alpha \in GL_2^+(\mathbb{Q})$, we have $([Y, X](F))_{\alpha} = X(F)_{\alpha}$. We know that for any element $g \in \mathcal{QM}$ and hence in particular for the element $F_{\alpha} \in \mathcal{QM}$, we have $[Y, X](g) = X(g)$. The result now follows from the definition of the action of X and Y in (4.15).

(b) The Lie action of \mathfrak{l}_1 on $\mathcal{Q}_{\sigma}(\Gamma)$ from part (a) induces an action of the universal enveloping algebra \mathfrak{h}_1 on $\mathcal{Q}_{\sigma}(\Gamma)$. In order to prove (4.16), it suffices to prove the result for the generators X and Y . We have:

$$\begin{aligned}
(X(F^1, F^2))_{\alpha} &= X((F^1, F^2)_{\alpha}) \\
&= X \left(\sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} F_{\beta\sigma}^1 \cdot (F_{\alpha\sigma^{-1}\beta^{-1}}^2 || \sigma\beta) \right) \\
&= \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} X(F_{\beta\sigma}^1) \cdot (F_{\alpha\sigma^{-1}\beta^{-1}}^2 || \sigma\beta) + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_{\beta\sigma}^1 \cdot X(F_{\alpha\sigma^{-1}\beta^{-1}}^2 || \sigma\beta) \\
&= \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} X(F_{\beta\sigma}^1) \cdot (F_{\alpha\sigma^{-1}\beta^{-1}}^2 || \sigma\beta) + \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} F_{\beta\sigma}^1 \cdot (X(F_{\alpha\sigma^{-1}\beta^{-1}}^2) || \sigma\beta) \\
&= (X(F^1), F^2)_{\alpha} + (F^1, X(F^2))_{\alpha}
\end{aligned} \tag{4.17}$$

In (4.17), we have used the fact that $\sigma\beta \in SL_2(\mathbb{Z})$ and hence $X(F_{\alpha\sigma^{-1}\beta^{-1}}^2 || \sigma\beta) = X(F_{\alpha\sigma^{-1}\beta^{-1}}^2) || \sigma\beta$. We can similarly verify the relation (4.16) for $Y \in \mathfrak{h}_1$. This proves the result. \square

Our next aim is to show that $\mathcal{Q}_\sigma(\Gamma)$ is a right $\mathcal{Q}(\Gamma)$ -module. Thereafter, we will consider the Hopf algebra \mathcal{H}_1 defined in Section 3.2 and show that there is a ‘‘Hopf action’’ of \mathcal{H}_1 on the right $\mathcal{Q}(\Gamma)$ -module $\mathcal{Q}_\sigma(\Gamma)$.

Proposition 4.4. *Let $\sigma \in SL_2(\mathbb{Z})$ and let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. Then, $\mathcal{Q}_\sigma(\Gamma)$ carries a right $\mathcal{Q}(\Gamma)$ -module structure defined by:*

$$(F^1 * F^2)_\alpha := \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_{\beta\sigma}^1 \cdot (F_{\alpha\sigma^{-1}\beta^{-1}}^2 | \beta) \quad (4.18)$$

for any $F^1 \in \mathcal{Q}_\sigma(\Gamma)$ and any $F^2 \in \mathcal{Q}(\Gamma)$.

Proof. We take $\gamma \in \Gamma$. Then, since $F^1 \in \mathcal{Q}_\sigma(\Gamma)$ and $F^2 \in \mathcal{Q}(\Gamma)$, we have:

$$F_{\gamma\beta\sigma}^1 = F_{\beta\sigma}^1 \quad F_{\alpha\sigma^{-1}\beta^{-1}\gamma^{-1}}^2 | \gamma\beta = F_{\alpha\sigma^{-1}\beta^{-1}}^2 | \gamma^{-1}\gamma\beta = F_{\alpha\sigma^{-1}\beta^{-1}}^2 | \beta \quad (4.19)$$

It follows that the sum in (4.18) is well defined, i.e., it does not depend on the choice of coset representatives for Γ in $GL_2^+(\mathbb{Q})$. Further, it is clear that $(F^1 * F^2)_{\gamma\alpha} = (F^1 * F^2)_\alpha$. In order to show that $F^1 * F^2 \in \mathcal{Q}_\sigma(\Gamma)$, it remains to show that $(F^1 * F^2)_{\alpha\gamma} = (F^1 * F^2)_\alpha || \sigma\gamma\sigma^{-1}$. By definition, we know that:

$$(F^1 * F^2)_{\alpha\gamma} = \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_{\beta\sigma}^1 \cdot (F_{\alpha\gamma\sigma^{-1}\beta^{-1}}^2 | \beta) \quad (4.20)$$

We now set $\delta = \beta\sigma\gamma^{-1}\sigma^{-1}$. This allows us to rewrite (4.20) as follows:

$$\begin{aligned} (F^1 * F^2)_{\alpha\gamma} &= \sum_{\delta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_{\delta\sigma\gamma}^1 \cdot (F_{\alpha\sigma^{-1}\delta^{-1}}^2 | \delta\sigma\gamma\sigma^{-1}) \\ &= \sum_{\delta \in \Gamma \backslash GL_2^+(\mathbb{Q})} (F_{\delta\sigma}^1 || \sigma\gamma\sigma^{-1}) \cdot ((F_{\alpha\sigma^{-1}\delta^{-1}}^2 | \delta) | \sigma\gamma\sigma^{-1}) \\ &= \left(\sum_{\delta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_{\delta\sigma}^1 \cdot (F_{\alpha\sigma^{-1}\delta^{-1}}^2 | \delta) \right) || \sigma\gamma\sigma^{-1} \\ &= (F^1 * F^2)_\alpha || \sigma\gamma\sigma^{-1} \end{aligned} \quad (4.21)$$

Hence, $(F^1 * F^2) \in \mathcal{Q}_\sigma(\Gamma)$. In order to show that $\mathcal{Q}_\sigma(\Gamma)$ is a right $\mathcal{Q}(\Gamma)$ -module, we need to check that $F^1 * (F^2 * F^3) = (F^1 * F^2) * F^3$ for any $F^1 \in \mathcal{Q}_\sigma(\Gamma)$ and any $F^2, F^3 \in \mathcal{Q}(\Gamma)$. For this, we note that:

$$(F^1 * F^2)_\alpha = \sum_{\alpha_2\alpha_1=\alpha} F_{\alpha_1}^1 \cdot (F_{\alpha_2}^2 | \alpha_1\sigma^{-1}) \quad \forall \alpha \in GL_2^+(\mathbb{Q}) \quad (4.22)$$

where the sum in (4.22) is taken over all pairs (α_1, α_2) such that $\alpha_2\alpha_1 = \alpha$ modulo the the following equivalence relation:

$$(\alpha_1, \alpha_2) \sim (\gamma\alpha_1, \alpha_2\gamma^{-1}) \quad \forall \gamma \in \Gamma \quad (4.23)$$

It follows that for any $\alpha \in GL_2^+(\mathbb{Q})$, we have:

$$((F^1 * F^2) * F^3)_\alpha = \sum_{\alpha_3\alpha_2\alpha_1=\alpha} F_{\alpha_1}^1 \cdot (F_{\alpha_2}^2 | \alpha_1\sigma^{-1}) \cdot (F_{\alpha_3}^3 | \alpha_2\alpha_1\sigma^{-1}) \quad (4.24)$$

where the sum in (4.24) is taken over all triples $(\alpha_1, \alpha_2, \alpha_3)$ such that $\alpha_3\alpha_2\alpha_1 = \alpha$ modulo the following equivalence relation:

$$(\alpha_1, \alpha_2, \alpha_3) \sim (\gamma\alpha_1, \gamma'\alpha_2\gamma'^{-1}, \alpha_3\gamma'^{-1}) \quad \forall \gamma, \gamma' \in \Gamma \quad (4.25)$$

On the other hand, we have:

$$\begin{aligned} (F^1 * (F^2 * F^3))_\alpha &= \sum_{\alpha'_2 \alpha_1 = \alpha} F^1_{\alpha_1} \cdot ((F^2 * F^3)_{\alpha'_2} | \alpha_1 \sigma^{-1}) \\ &= \sum_{\alpha_3 \alpha_2 \alpha_1 = \alpha} F^1_{\alpha_1} \cdot (F^2_{\alpha_2} | \alpha_1 \sigma^{-1}) \cdot (F^3_{\alpha_3} | \alpha_2 \alpha_1 \sigma^{-1}) \end{aligned} \quad (4.26)$$

Again, we see that the sum in (4.26) is taken over all triples $(\alpha_1, \alpha_2, \alpha_3)$ such that $\alpha_3 \alpha_2 \alpha_1 = \alpha$ modulo the equivalence relation in (4.25). From (4.24) and (4.26), it follows that $(F^1 * (F^2 * F^3))_\alpha = ((F^1 * F^2) * F^3)_\alpha$. This proves the result. \square

We are now ready to describe the action of the Hopf algebra \mathcal{H}_1 on $\mathcal{Q}_\sigma(\Gamma)$. From Section 3.2, we know that \mathcal{H}_1 is generated by $X, Y, \{\delta_n\}_{n \geq 1}$ which satisfy the relations (3.44), (3.45), (3.46).

Proposition 4.5. *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and choose some $\sigma \in SL_2(\mathbb{Z})$.*

(a) *The collection of σ -twisted quasimodular Hecke operators of level Γ can be made into an \mathcal{H}_1 -module as follows; for any $F \in \mathcal{Q}_\sigma(\Gamma)$ and $\alpha \in GL_2^+(\mathbb{Q})$:*

$$X(F)_\alpha := X(F_\alpha) \quad Y(F)_\alpha := Y(F_\alpha) \quad \delta_n(F)_\alpha := X^{n-1}(\mu_{\alpha\sigma^{-1}}) \cdot F_\alpha \quad \forall n \geq 1 \quad (4.27)$$

(b) *The Hopf algebra \mathcal{H}_1 has a ‘‘Hopf action’’ on the right $\mathcal{Q}(\Gamma)$ -module $\mathcal{Q}_\sigma(\Gamma)$; in other words, for any $F^1 \in \mathcal{Q}_\sigma(\Gamma)$ and any $F^2 \in \mathcal{Q}(\Gamma)$, we have:*

$$h(F^1 * F^2) = \sum h_{(1)}(F^1) * h_{(2)}(F^2) \quad \forall h \in \mathcal{H}_1 \quad (4.28)$$

where the coproduct $\Delta : \mathcal{H}_1 \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_1$ is given by $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for each $h \in \mathcal{H}_1$.

Proof. (a) For any $F \in \mathcal{Q}_\sigma(\Gamma)$, we have already checked in the proof of Proposition 4.3 that $X(F), Y(F) \in \mathcal{Q}_\sigma(\Gamma)$. Further, from (3.55), we know that for any $\alpha \in GL_2^+(\mathbb{Q})$ and $\gamma \in \Gamma$, we have:

$$\begin{aligned} \mu_{\gamma\alpha\sigma^{-1}} &= \mu_\gamma | \alpha \sigma^{-1} + \mu_{\alpha\sigma^{-1}} = \mu_{\alpha\sigma^{-1}} \\ \mu_{\alpha\gamma\sigma^{-1}} &= \mu_{\alpha\sigma^{-1}} | \sigma \gamma \sigma^{-1} + \mu_{\sigma\gamma\sigma^{-1}} = \mu_{\alpha\sigma^{-1}} | \sigma \gamma \sigma^{-1} \end{aligned} \quad (4.29)$$

Hence, for any $F \in \mathcal{Q}_\sigma(\Gamma)$, we have:

$$\begin{aligned} \delta_n(F)_{\gamma\alpha} &= X^{n-1}(\mu_{\gamma\alpha\sigma^{-1}}) \cdot F_{\gamma\alpha} = X^{n-1}(\mu_{\alpha\sigma^{-1}}) \cdot F_\alpha = \delta_n(F)_\alpha \\ \delta_n(F)_{\alpha\gamma} &= X^{n-1}(\mu_{\alpha\gamma\sigma^{-1}}) \cdot F_{\alpha\gamma} = X^{n-1}(\mu_{\alpha\sigma^{-1}} | \sigma \gamma \sigma^{-1}) \cdot (F_\alpha | \sigma \gamma \sigma^{-1}) = \delta_n(F)_\alpha | \sigma \gamma \sigma^{-1} \end{aligned} \quad (4.30)$$

Hence, $\delta_n(F) \in \mathcal{Q}_\sigma(\Gamma)$. In order to show that there is an action of the Lie algebra \mathcal{L}_1 (and hence of its universal enveloping algebra \mathcal{H}_1) on $\mathcal{Q}_\sigma(\Gamma)$, it remains to check the commutator relations (3.44) between the operators X, Y and δ_n acting on $\mathcal{Q}_\sigma(\Gamma)$. We have already checked that $[Y, X] = X$ in the proof of Proposition 4.3. Since X is a derivation on \mathcal{QM} and $\delta_n(F)_\alpha = X^{n-1}(\mu_{\alpha\sigma^{-1}}) \cdot F_\alpha$, we have:

$$\begin{aligned} [X, \delta_n](F)_\alpha &= X(X^{n-1}(\mu_{\alpha\sigma^{-1}}) \cdot F_\alpha) - X^{n-1}(\mu_{\alpha\sigma^{-1}}) \cdot X(F_\alpha) \\ &= X(X^{n-1}(\mu_{\alpha\sigma^{-1}})) \cdot F_\alpha = X^n(\mu_{\alpha\sigma^{-1}}) \cdot F_\alpha = \delta_{n+1}(F)_\alpha \end{aligned} \quad (4.31)$$

Similarly, since $\mu_{\alpha\sigma^{-1}} \in \mathcal{M} \subseteq \mathcal{QM}$ is of weight 2 and Y is a derivation on \mathcal{QM} , we have:

$$\begin{aligned} [Y, \delta_n](F)_\alpha &= Y(X^{n-1}(\mu_{\alpha\sigma^{-1}}) \cdot F_\alpha) - X^{n-1}(\mu_{\alpha\sigma^{-1}}) \cdot Y(F_\alpha) \\ &= Y(X^{n-1}(\mu_{\alpha\sigma^{-1}})) \cdot F_\alpha = nX^{n-1}(\mu_{\alpha\sigma^{-1}}) \cdot F_\alpha = n\delta_n(F)_\alpha \end{aligned} \quad (4.32)$$

Finally, we can verify easily that $[\delta_k, \delta_l] = 0$ for any $k, l \geq 1$.

(b) In order to prove (4.28), it is enough to check this equality for the generators X, Y and $\delta_1 \in \mathcal{H}_1$. For $F^1 \in \mathcal{Q}_\sigma(\Gamma)$, $F^2 \in \mathcal{Q}(\Gamma)$ and $\alpha \in GL_2^+(\mathbb{Q})$, we have:

$$\begin{aligned}
& (X(F^1 * F^2))_\alpha = X((F^1 * F^2)_\alpha) \\
&= \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} X(F_{\beta\sigma}^1 \cdot (F_{\alpha\sigma^{-1}\beta^{-1}}^2 | \beta)) \\
&= \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} X(F_{\beta\sigma}^1) \cdot (F_{\alpha\sigma^{-1}\beta^{-1}}^2 | \beta) + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_{\beta\sigma}^1 \cdot X(F_{\alpha\sigma^{-1}\beta^{-1}}^2 | \beta) \\
&= (X(F^1) * F^2)_\alpha + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_{\beta\sigma}^1 \cdot X(F_{\alpha\sigma^{-1}\beta^{-1}}^2 | \beta) - \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_{\beta\sigma}^1 \cdot (\mu_{\beta^{-1}} | \beta) \cdot Y(F_{\alpha\sigma^{-1}\beta^{-1}}^2 | \beta) \quad (4.33) \\
&= (X(F^1) * F^2)_\alpha + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_{\beta\sigma}^1 \cdot X(F_{\alpha\sigma^{-1}\beta^{-1}}^2 | \beta) + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_{\beta\sigma}^1 \cdot \mu_\beta \cdot Y(F_{\alpha\sigma^{-1}\beta^{-1}}^2 | \beta) \\
&= (X(F^1) * F^2)_\alpha + (F^1 * X(F^2))_\alpha + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} \delta_1(F)_{\beta\sigma} \cdot Y(F^2)_{\alpha\sigma^{-1}\beta^{-1}} | \beta \\
&= (X(F^1) * F^2)_\alpha + (F^1 * X(F^2))_\alpha + (\delta_1(F^1) * Y(F^2))_\alpha
\end{aligned}$$

In (4.33) above, we have used the fact that $0 = \mu_{\beta^{-1}\beta} = \mu_{\beta^{-1}} | \beta + \mu_\beta$. For $\alpha, \beta \in GL_2^+(\mathbb{Q})$, it follows from (3.55) that

$$\mu_{\alpha\sigma^{-1}} = \mu_{\alpha\sigma^{-1}\beta^{-1}\beta} = \mu_{\alpha\sigma^{-1}\beta^{-1}} | \beta + \mu_\beta \quad (4.34)$$

Since $F^2 \in \mathcal{Q}(\Gamma)$ we know from (3.56) that $\delta_1(F^2)_{\alpha\sigma^{-1}\beta^{-1}} = \mu_{\alpha\sigma^{-1}\beta^{-1}} \cdot F_{\alpha\sigma^{-1}\beta^{-1}}^2$. Combining with (4.34), we have:

$$\begin{aligned}
& \delta_1((F^1 * F^2))_\alpha = \mu_{\alpha\sigma^{-1}} \cdot (F^1 * F^2)_\alpha = \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} \mu_{\alpha\sigma^{-1}} \cdot (F_{\beta\sigma}^1 \cdot (F_{\alpha\sigma^{-1}\beta^{-1}}^2 | \beta)) \\
&= \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} (\mu_\beta \cdot F_{\beta\sigma}^1) \cdot (F_{\alpha\sigma^{-1}\beta^{-1}}^2 | \beta) + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_{\beta\sigma}^1 \cdot (\mu_{\alpha\sigma^{-1}\beta^{-1}} \cdot F_{\alpha\sigma^{-1}\beta^{-1}}^2 | \beta) \quad (4.35) \\
&= (\delta_1(F^1) * F^2)_\alpha + (F^1 * \delta_1(F^2))_\alpha
\end{aligned}$$

Finally, from the definition of Y , it is easy to show that $(Y(F^1 * F^2))_\alpha = (Y(F^1) * F^2)_\alpha + (F^1 * Y(F^2))_\alpha$. \square

4.2 The operators $X_\tau : \mathcal{Q}_\sigma(\Gamma) \longrightarrow \mathcal{Q}_{\tau\sigma}(\Gamma)$ and Hopf action

Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup and choose some $\sigma \in SL_2(\mathbb{Z})$. In Section 4.1, we have only considered operators X, Y and $\{\delta_n\}_{n \geq 1}$ that are endomorphisms of $\mathcal{Q}_\sigma(\Gamma)$. In this section, we will define an operator

$$X_\tau : \mathcal{Q}_\sigma(\Gamma) \longrightarrow \mathcal{Q}_{\tau\sigma}(\Gamma) \quad (4.36)$$

for $\tau \in SL_2(\mathbb{Z})$. In particular, we consider the commuting family $\left\{ \rho_n := \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\}_{n \in \mathbb{Z}}$ of matrices in $SL_2(\mathbb{Z})$ and write $\sigma(n) := \rho_n \cdot \sigma$. Then, we have operators:

$$X_{\rho_n} : \mathcal{Q}_{\sigma(m)}(\Gamma) \longrightarrow \mathcal{Q}_{\sigma(m+n)}(\Gamma) \quad \forall m, n \in \mathbb{Z} \quad (4.37)$$

acting “between the levels” of the graded module $\mathbb{Q}_\sigma(\Gamma) := \bigoplus_{m \in \mathbb{Z}} \mathcal{Q}_{\sigma(m)}(\Gamma)$. We already know that $\mathcal{Q}_\sigma(\Gamma)$ carries an action of the Hopf algebra \mathfrak{h}_1 . Further, \mathfrak{h}_1 has a Hopf action on the pairing on $\mathcal{Q}_\sigma(\Gamma)$ in the sense of Proposition 4.3. We will now show that \mathfrak{h}_1 can be naturally embedded into a larger Hopf algebra $\mathfrak{h}_\mathbb{Z}$ acting on

$\mathbb{Q}_\sigma(\Gamma)$ that incorporates the operators X_{ρ_n} in (4.37). Finally, we will show that the pairing on $\mathcal{Q}_\sigma(\Gamma)$ can be extended to a pairing:

$$(_, _) : \mathcal{Q}_{\sigma(m)}(\Gamma) \otimes \mathcal{Q}_{\sigma(n)}(\Gamma) \longrightarrow \mathcal{Q}_{\sigma(m+n)}(\Gamma) \quad \forall m, n \in \mathbb{Z} \quad (4.38)$$

This gives us a pairing on $\mathbb{Q}_\sigma(\Gamma)$ and we prove that this pairing carries a Hopf action of $\mathfrak{h}_\mathbb{Z}$. We start by defining the operators X_τ mentioned in (4.36).

Proposition 4.6. (a) Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and choose $\sigma \in SL_2(\mathbb{Z})$.

(a) For each $\tau \in SL_2(\mathbb{Z})$, we have a morphism:

$$X_\tau : \mathcal{Q}_\sigma(\Gamma) \longrightarrow \mathcal{Q}_{\tau\sigma}(\Gamma) \quad X_\tau(F)_\alpha := X(F_\alpha) \|\tau^{-1} \quad \forall F \in \mathcal{Q}_\sigma(\Gamma), \alpha \in GL_2^+(\mathbb{Q}) \quad (4.39)$$

(b) Let $\tau_1, \tau_2 \in SL_2(\mathbb{Z})$ be two matrices such that $\tau_1\tau_2 = \tau_2\tau_1$. Then, the commutator $[X_{\tau_1}, X_{\tau_2}] = 0$.

Proof. (a) We choose any $F \in \mathcal{Q}_\sigma(\Gamma)$. From (4.39), it is clear that $X_\tau(F)_{\gamma\alpha} = X_\tau(F)_\alpha$ for any $\gamma \in \Gamma$ and $\alpha \in GL_2^+(\mathbb{Q})$. Further, we note that:

$$\begin{aligned} X_\tau(F)_{\alpha\gamma} &= X(F_{\alpha\gamma}) \|\tau^{-1} = X(F_\alpha \|\sigma\gamma\sigma^{-1}) \|\tau^{-1} \\ &= X(F_\alpha \|\tau^{-1}) \|\tau\sigma\gamma\sigma^{-1}\tau^{-1} \\ &= X_\tau(F_\alpha) \|\tau\sigma\gamma\sigma^{-1}\tau^{-1} \end{aligned} \quad (4.40)$$

It follows from (4.40) that $X_\tau(F) \in \mathcal{Q}_{\tau\sigma}(\Gamma)$ for any $F \in \mathcal{Q}_\sigma(\Gamma)$.

(b) Since τ_1 and τ_2 commute, both $X_{\tau_1}X_{\tau_2}$ and $X_{\tau_2}X_{\tau_1}$ are operators from $\mathcal{Q}_\sigma(\Gamma)$ to $\mathcal{Q}_{\tau_1\tau_2\sigma}(\Gamma) = \mathcal{Q}_{\tau_2\tau_1\sigma}(\Gamma)$. For any $F \in \mathcal{Q}_\sigma(\Gamma)$, we have $(\forall \alpha \in GL_2^+(\mathbb{Q}))$:

$$(X_{\tau_1}X_{\tau_2}(F))_\alpha = X(X_{\tau_2}(F)_\alpha) \|\tau_1^{-1} = X^2(F_\alpha) \|\tau_2^{-1}\tau_1^{-1} = X^2(F_\alpha) \|\tau_1^{-1}\tau_2^{-1} = (X_{\tau_2}X_{\tau_1}(F))_\alpha \quad (4.41)$$

This proves the result. \square

As mentioned before, we now consider the commuting family $\left\{ \rho_n := \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\}_{n \in \mathbb{Z}}$ of matrices in $SL_2(\mathbb{Z})$ and set $\sigma(n) := \rho_n \cdot \sigma$ for any $\sigma \in SL_2(\mathbb{Z})$. We want to define a pairing on the graded module $\mathbb{Q}_\sigma(\Gamma) = \bigoplus_{m \in \mathbb{Z}} \mathcal{Q}_{\sigma(m)}(\Gamma)$ that extends the pairing on $\mathcal{Q}_\sigma(\Gamma)$. In fact, we will prove a more general result.

Proposition 4.7. Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and choose $\sigma \in SL_2(\mathbb{Z})$. Let $\tau_1, \tau_2 \in SL_2(\mathbb{Z})$ be two matrices such that $\tau_1\tau_2 = \tau_2\tau_1$. Then, there exists a pairing

$$(_, _) : \mathcal{Q}_{\tau_1\sigma}(\Gamma) \otimes \mathcal{Q}_{\tau_2\sigma}(\Gamma) \longrightarrow \mathcal{Q}_{\tau_1\tau_2\sigma}(\Gamma) \quad (4.42)$$

defined as follows: for any $F^1 \in \mathcal{Q}_{\tau_1\sigma}(\Gamma)$ and any $F^2 \in \mathcal{Q}_{\tau_2\sigma}(\Gamma)$, we set:

$$(F^1, F^2)_\alpha := \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} (F_{\beta\sigma}^1 \|\tau_2^{-1}) \cdot (F_{\alpha\sigma^{-1}\beta^{-1}}^2 \|\tau_2\sigma\beta\tau_1^{-1}\tau_2^{-1}) \quad \forall \alpha \in GL_2^+(\mathbb{Q}) \quad (4.43)$$

In particular, when $\tau_1 = \tau_2 = 1$, the pairing in (4.43) reduces to the pairing on $\mathcal{Q}_\sigma(\Gamma)$ defined in (4.11).

Proof. We choose some $\gamma \in \Gamma$. Then, for any $\alpha \in GL_2^+(\mathbb{Q})$, $\beta \in SL_2(\mathbb{Z})$, we have $F_{\gamma\beta\sigma}^1 = F_{\beta\sigma}^1$ and:

$$(F_{\alpha\sigma^{-1}\beta^{-1}\gamma^{-1}}^2 || \tau_2 \sigma \gamma \beta \tau_1^{-1} \tau_2^{-1}) = (F_{\alpha\sigma^{-1}\beta^{-1}}^2 || \tau_2 \sigma \gamma^{-1} \sigma^{-1} \tau_2^{-1} \tau_2 \sigma \gamma \beta \tau_1^{-1} \tau_2^{-1}) = (F_{\alpha\sigma^{-1}\beta^{-1}}^2 || \tau_2 \sigma \beta \tau_1^{-1} \tau_2^{-1})$$

It follows that the sum in (4.43) is well defined, i.e., independent of the choice of coset representatives of Γ in $SL_2(\mathbb{Z})$. Additionally, we have:

$$(F^1, F^2)_{\alpha\gamma} := \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} (F_{\delta\sigma}^1 || \tau_2^{-1}) \cdot (F_{\alpha\gamma\sigma^{-1}\beta^{-1}}^2 || \tau_2 \sigma \beta \tau_1^{-1} \tau_2^{-1}) \quad (4.44)$$

We now set $\delta = \beta\sigma\gamma^{-1}\sigma^{-1}$. Since $F^1 \in \mathcal{Q}_{\tau_1\sigma}(\Gamma)$, we know that $F_{\delta\sigma\gamma}^1 = F_{\delta\sigma}^1 || \tau_1 \sigma \gamma \sigma^{-1} \tau_1^{-1}$. Then, we can rewrite the expression in (4.44) as follows:

$$\begin{aligned} (F^1, F^2)_{\alpha\gamma} &= \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} (F_{\delta\sigma\gamma}^1 || \tau_2^{-1}) \cdot (F_{\alpha\sigma^{-1}\delta^{-1}}^2 || \tau_2 \sigma \delta \sigma \gamma \sigma^{-1} \tau_1^{-1} \tau_2^{-1}) \\ &= \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} (F_{\delta\sigma}^1 || \tau_1 \sigma \gamma \sigma^{-1} \tau_1^{-1} \tau_2^{-1}) \cdot (F_{\alpha\sigma^{-1}\delta^{-1}}^2 || \tau_2 \sigma \delta \sigma \gamma \sigma^{-1} \tau_1^{-1} \tau_2^{-1}) \\ &= \left(\sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} (F_{\delta\sigma}^1 || \tau_2^{-1}) \cdot (F_{\alpha\sigma^{-1}\delta^{-1}}^2 || \tau_2 \sigma \delta \tau_1^{-1} \tau_2^{-1}) \right) || \tau_1 \tau_2 \sigma \gamma \sigma^{-1} \tau_1^{-1} \tau_2^{-1} \\ &= (F^1, F^2)_{\alpha} || \tau_1 \tau_2 \sigma \gamma \sigma^{-1} \tau_1^{-1} \tau_2^{-1} \end{aligned} \quad (4.45)$$

From (4.45) it follows that $(F^1, F^2) \in \mathcal{Q}_{\tau_1\tau_2\sigma}(\Gamma)$. □

In particular, it follows from the pairing in (4.42) that for any $m, n \in \mathbb{Z}$, we have a pairing

$$(_, _) : \mathcal{Q}_{\sigma(m)}(\Gamma) \otimes \mathcal{Q}_{\sigma(n)}(\Gamma) \longrightarrow \mathcal{Q}_{\sigma(m+n)}(\Gamma) \quad (4.46)$$

It is clear that (4.46) induces a pairing on $\mathbb{Q}_{\sigma}(\Gamma) = \bigoplus_{m \in \mathbb{Z}} \mathcal{Q}_{\sigma(m)}(\Gamma)$ for each $\sigma \in SL_2(\mathbb{Z})$. We will now define operators $\{X_n\}_{n \in \mathbb{Z}}$ and Z on $\mathbb{Q}_{\sigma}(\Gamma)$. For each $n \in \mathbb{Z}$, the operator $X_n : \mathbb{Q}_{\sigma}(\Gamma) \longrightarrow \mathbb{Q}_{\sigma}(\Gamma)$ is induced by the collection of operators:

$$X_n^m := X_{\rho_n} : \mathcal{Q}_{\sigma(m)}(\Gamma) \longrightarrow \mathcal{Q}_{\sigma(m+n)}(\Gamma) \quad \forall m \in \mathbb{Z} \quad (4.47)$$

where, as mentioned before, $\rho_n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Then, $X_n : \mathbb{Q}_{\sigma}(\Gamma) \longrightarrow \mathbb{Q}_{\sigma}(\Gamma)$ is an operator of homogeneous degree n on the graded module $\mathbb{Q}_{\sigma}(\Gamma)$. We also consider:

$$Z : \mathcal{Q}_{\sigma(m)}(\Gamma) \longrightarrow \mathcal{Q}_{\sigma(m)}(\Gamma) \quad Z(F)_{\alpha} := mF_{\alpha} + Y(F_{\alpha}) \quad \forall F \in \mathcal{Q}_{\sigma(m)}(\Gamma), \alpha \in GL_2^+(\mathbb{Q}) \quad (4.48)$$

This induces an operator $Z : \mathbb{Q}_{\sigma}(\Gamma) \longrightarrow \mathbb{Q}_{\sigma}(\Gamma)$ of homogeneous degree 0 on the graded module $\mathbb{Q}_{\sigma}(\Gamma)$. We will now show that $\mathbb{Q}_{\sigma}(\Gamma)$ is acted upon by a certain Lie algebra $\mathfrak{l}_{\mathbb{Z}}$ such that the Lie action incorporates the operators $\{X_n\}_{n \in \mathbb{Z}}$ and Z mentioned above. We define $\mathfrak{l}_{\mathbb{Z}}$ to be the Lie algebra with generators $\{Z, X_n | n \in \mathbb{Z}\}$ satisfying the following commutator relations:

$$[Z, X_n] = (n+1)X_n \quad [X_n, X_{n'}] = 0 \quad \forall n, n' \in \mathbb{Z} \quad (4.49)$$

In particular, we note that $[Z, X_0] = X_0$. It follows that the Lie algebra $\mathfrak{l}_{\mathbb{Z}}$ contains the Lie algebra \mathfrak{l}_1 defined in (3.57). We now describe the action of $\mathfrak{l}_{\mathbb{Z}}$ on $\mathbb{Q}_{\sigma}(\Gamma)$.

Proposition 4.8. *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and let $\sigma \in SL_2(\mathbb{Z})$. Then, the Lie algebra $\mathfrak{l}_{\mathbb{Z}}$ has a Lie action on $\mathcal{Q}_{\sigma}(\Gamma)$.*

Proof. We need to check that $[Z, X_n] = (n+1)X_n$ and $[X_n, X_{n'}] = 0, \forall n, n' \in \mathbb{Z}$ for the operators $\{Z, X_n | n \in \mathbb{Z}\}$ on $\mathcal{Q}_{\sigma}(\Gamma)$. From part (b) of Proposition 4.6, we know that $[X_n, X_{n'}] = 0$. From (4.47) and (4.48), it is clear that in order to show that $[Z, X_n] = (n+1)X_n$, we need to check that $[Z, X_n^m] = (n+1)X_n^m : \mathcal{Q}_{\sigma(m)}(\Gamma) \rightarrow \mathcal{Q}_{\sigma(m+n)}(\Gamma)$ for any given $m \in \mathbb{Z}$. For any $F \in \mathcal{Q}_{\sigma(m)}(\Gamma)$ and any $\alpha \in GL_2^+(\mathbb{Q})$, we now check that:

$$\begin{aligned} (ZX_n^m(F))_{\alpha} &= (n+m)X_n^m(F)_{\alpha} + Y(X_n^m(F)_{\alpha}) = (n+m)X(F_{\alpha})||\rho_n^{-1} + YX(F_{\alpha})||\rho_n^{-1} \\ (X_n^m Z(F))_{\alpha} &= X(Z(F)_{\alpha})||\rho_n^{-1} = mX(F_{\alpha})||\rho_n^{-1} + XY(F_{\alpha})||\rho_n^{-1} \end{aligned} \quad (4.50)$$

Combining (4.50) with the fact that $[Y, X] = X$, it follows that $[Z, X_n^m] = (n+1)X_n^m$ for each $m \in \mathbb{Z}$. Hence, the result follows. \square

We now consider the Hopf algebra $\mathfrak{h}_{\mathbb{Z}}$ that is the universal enveloping algebra of the Lie algebra $\mathfrak{l}_{\mathbb{Z}}$. Accordingly, the coproduct Δ on $\mathfrak{h}_{\mathbb{Z}}$ is given by:

$$\Delta(X_n) = X_n \otimes 1 + 1 \otimes X_n \quad \Delta(Z) = Z \otimes 1 + 1 \otimes Z \quad \forall n \in \mathbb{Z} \quad (4.51)$$

It is clear that $\mathfrak{h}_{\mathbb{Z}}$ contains the Hopf algebra \mathfrak{h}_1 , the universal enveloping algebra of \mathfrak{l}_1 . From Proposition 4.3, we know that \mathfrak{h}_1 has a Hopf action on the pairing on $\mathcal{Q}_{\sigma}(\Gamma)$. We want to show that $\mathfrak{h}_{\mathbb{Z}}$ has a Hopf action on the pairing on $\mathcal{Q}_{\sigma}(\Gamma)$. For this, we prove the following Lemma.

Lemma 4.9. *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and let $\sigma \in SL_2(\mathbb{Z})$. Let $\tau_1, \tau_2, \tau_3 \in SL_2(\mathbb{Z})$ be three matrices such that $\tau_i \tau_j = \tau_j \tau_i, \forall i, j \in \{1, 2, 3\}$. Then, for any $F^1 \in \mathcal{Q}_{\tau_1 \sigma}(\Gamma)$, $F^2 \in \mathcal{Q}_{\tau_2 \sigma}(\Gamma)$, we have:*

$$X_{\tau_3}(F^1, F^2) = (X_{\tau_3}(F^1), F^2) + (F^1, X_{\tau_3}(F^2)) \quad (4.52)$$

Proof. Consider any $\alpha \in GL_2^+(\mathbb{Q})$. Then, from the definition of X_{τ_3} , it follows that

$$\begin{aligned} X_{\tau_3}(F^1, F^2)_{\alpha} &= \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} X((F_{\beta \sigma}^1 || \tau_2^{-1}) \cdot (F_{\alpha \sigma^{-1} \beta^{-1}}^2 || \tau_2 \sigma \beta \tau_1^{-1} \tau_2^{-1})) || \tau_3^{-1} \\ &= \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} (X(F_{\beta \sigma}^1 || \tau_2^{-1} \tau_3^{-1}) \cdot (F_{\alpha \sigma^{-1} \beta^{-1}}^2 || \tau_2 \sigma \beta \tau_1^{-1} \tau_2^{-1} \tau_3^{-1})) \\ &\quad + \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} (F_{\beta \sigma}^1 || \tau_2^{-1} \tau_3^{-1}) \cdot (X(F_{\alpha \sigma^{-1} \beta^{-1}}^2 || \tau_2 \sigma \beta \tau_1^{-1} \tau_2^{-1} \tau_3^{-1})) \end{aligned} \quad (4.53)$$

Since $F^1 \in \mathcal{Q}_{\tau_1 \sigma}(\Gamma)$, it follows that $X_{\tau_3}(F^1) \in \mathcal{Q}_{\tau_1 \tau_3 \sigma}(\Gamma)$. Similarly, we see that $X_{\tau_3}(F^2) \in \mathcal{Q}_{\tau_2 \tau_3 \sigma}(\Gamma)$. It follows that:

$$\begin{aligned} (X_{\tau_3}(F^1), F^2)_{\alpha} &= \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} (X_{\tau_3}(F^1)_{\beta \sigma} || \tau_2^{-1}) \cdot (F_{\alpha \sigma^{-1} \beta^{-1}}^2 || \tau_2 \sigma \beta \tau_1^{-1} \tau_2^{-1} \tau_3^{-1}) \\ &= \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} (X(F_{\beta \sigma}^1 || \tau_2^{-1} \tau_3^{-1}) \cdot (F_{\alpha \sigma^{-1} \beta^{-1}}^2 || \tau_2 \sigma \beta \tau_1^{-1} \tau_2^{-1} \tau_3^{-1})) \\ (F^1, X_{\tau_3}(F^2))_{\alpha} &= \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} (F_{\beta \sigma}^1 || \tau_2^{-1} \tau_3^{-1}) \cdot (X_{\tau_3}(F^2)_{\alpha \sigma^{-1} \beta^{-1}} || \tau_2 \tau_3 \sigma \beta \tau_1^{-1} \tau_2^{-1} \tau_3^{-1}) \\ &= \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} (F_{\beta \sigma}^1 || \tau_2^{-1} \tau_3^{-1}) \cdot (X(F_{\alpha \sigma^{-1} \beta^{-1}}^2 || \tau_3^{-1} \tau_2 \tau_3 \sigma \beta \tau_1^{-1} \tau_2^{-1} \tau_3^{-1})) \\ &= \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} (F_{\beta \sigma}^1 || \tau_2^{-1} \tau_3^{-1}) \cdot (X(F_{\alpha \sigma^{-1} \beta^{-1}}^2 || \tau_2 \sigma \beta \tau_1^{-1} \tau_2^{-1} \tau_3^{-1})) \end{aligned} \quad (4.54)$$

Comparing (4.53) and (4.54), the result of (4.52) follows. \square

As a special case of Lemma 4.9, it follows that for any $F^1 \in \mathcal{Q}_{\sigma(m)}(\Gamma)$ and $F^2 \in \mathcal{Q}_{\sigma(m')}(\Gamma)$, we have:

$$X_{\rho_n}(F^1, F^2) = X_n(F^1, F^2) = (X_n(F^1), F^2) + (F^1, X_n(F^2)) \quad \forall n \in \mathbb{Z} \quad (4.55)$$

We conclude by showing that $\mathfrak{h}_{\mathbb{Z}}$ has a Hopf action on the pairing on $\mathbb{Q}_{\sigma}(\Gamma)$.

Proposition 4.10. *Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and let $\sigma \in SL_2(\mathbb{Z})$. Then, the Hopf algebra $\mathfrak{h}_{\mathbb{Z}}$ has a Hopf action on the pairing on $\mathbb{Q}_{\sigma}(\Gamma)$. In other words, for $F^1, F^2 \in \mathbb{Q}_{\sigma}(\Gamma)$, we have*

$$h(F^1, F^2) = \sum (h_{(1)}(F^1), h_{(2)}(F^2)) \quad (4.56)$$

where the coproduct $\Delta : \mathfrak{h}_{\mathbb{Z}} \rightarrow \mathfrak{h}_{\mathbb{Z}} \otimes \mathfrak{h}_{\mathbb{Z}}$ is defined by setting $\Delta(h) := \sum h_{(1)} \otimes h_{(2)}$ for each $h \in \mathfrak{h}_{\mathbb{Z}}$.

Proof. It suffices to prove the result in the case where $F^1 \in \mathcal{Q}_{\sigma(m)}(\Gamma)$, $F^2 \in \mathcal{Q}_{\sigma(m')}(\Gamma)$ for some $m, m' \in \mathbb{Z}$. Further, it suffices to prove the relation (4.56) for the generators $\{Z, X_n | n \in \mathbb{Z}\}$ of the Hopf algebra $\mathfrak{h}_{\mathbb{Z}}$. For the generators X_n , $n \in \mathbb{Z}$, this is already the result of (4.55) which follows from Lemma 4.9. Since $\Delta(Z) = Z \otimes 1 + 1 \otimes Z$, it remains to show that

$$Z(F^1, F^2) = (Z(F^1), F^2) + (F^1, Z(F^2)) \quad \forall F^1 \in \mathcal{Q}_{\sigma(m)}(\Gamma), F^2 \in \mathcal{Q}_{\sigma(m')}(\Gamma) \quad (4.57)$$

By the definition of the pairing on $\mathbb{Q}_{\sigma}(\Gamma)$, we know that $(F^1, F^2) \in \mathcal{Q}_{\sigma(m+m')}(\Gamma)$. Then, for any $\alpha \in GL_2^+(\mathbb{Q})$, we have:

$$\begin{aligned} Z(F^1, F^2)_{\alpha} &= (m + m')(F^1, F^2)_{\alpha} + Y(F^1, F^2)_{\alpha} \\ &= (m + m') \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} ((F^1_{\beta\sigma} || \rho_{m'}^{-1}) \cdot (F^2_{\alpha\sigma^{-1}\beta^{-1}} || \rho_{m'}\sigma\beta\rho_m^{-1}\rho_{m'}^{-1})) \\ &\quad + \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} Y((F^1_{\beta\sigma} || \rho_{m'}^{-1}) \cdot (F^2_{\alpha\sigma^{-1}\beta^{-1}} || \rho_{m'}\sigma\beta\rho_m^{-1}\rho_{m'}^{-1})) \\ &= \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} ((mF^1_{\beta\sigma} + Y(F^1_{\beta\sigma})) || \rho_{m'}^{-1}) \cdot (F^2_{\alpha\sigma^{-1}\beta^{-1}} || \rho_{m'}\sigma\beta\rho_m^{-1}\rho_{m'}^{-1}) \\ &\quad + \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} (F^1_{\beta\sigma} || \rho_{m'}^{-1}) \cdot ((m'F^2_{\alpha\sigma^{-1}\beta^{-1}} + Y(F^2_{\alpha\sigma^{-1}\beta^{-1}})) || \rho_{m'}\sigma\beta\rho_m^{-1}\rho_{m'}^{-1}) \\ &= \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} (Z(F^1)_{\beta\sigma} || \rho_{m'}^{-1}) \cdot (F^2_{\alpha\sigma^{-1}\beta^{-1}} || \rho_{m'}\sigma\beta\rho_m^{-1}\rho_{m'}^{-1}) \\ &\quad + \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} (F^1_{\beta\sigma} || \rho_{m'}^{-1}) \cdot (Z(F^2)_{\alpha\sigma^{-1}\beta^{-1}} || \rho_{m'}\sigma\beta\rho_m^{-1}\rho_{m'}^{-1}) \\ &= (Z(F^1), F^2)_{\alpha} + (F^1, Z(F^2))_{\alpha} \end{aligned} \quad (4.58)$$

□

References

- [1] A. Banerjee, Action de Hopf sur les opérateurs de Hecke modulaires tordus, *Journal of Noncommutative Geometry*, (To appear).
- [2] A. Connes, H. Moscovici, Hopf algebras, cyclic cohomology and the transverse index theorem, *Comm. Math. Phys.*, **198**, (1998), 199–246.
- [3] A. Connes, H. Moscovici, Modular Hecke algebras and their Hopf symmetry. *Mosc. Math. J.* **4** (2004), no. 1, 67–109.

- [4] A. Connes, H. Moscovici, Rankin-Cohen brackets and the Hopf algebra of transverse geometry. *Mosc. Math. J.* **4** (2004), no. 1, 111–130.
- [5] M. Kaneko, D. Zagier, A generalized Jacobi theta function and quasimodular forms. The moduli space of curves (Texel Island, 1994), 165–172, *Progr. Math.*, **129**, Birkhäuser Boston, Boston, MA, 1995.
- [6] E. Royer, Quasimodular forms: an introduction. *Ann. Math. Blaise Pascal* **19** (2012), no. 2, 297–306.
- [7] D. Zagier, Elliptic modular forms and their applications. The 1-2-3 of modular forms, 1–103, *Universitext*, Springer, Berlin, 2008.